MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

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CORRIGENDA.

VOL. II.

p. 84, equation (14), *for* $(x^2+y^2)^2$ *read* $(x^2-y^2)^2$;
p. 95, equation (1), *for* dy *read* dy'.

VOL. III.

p. 32, line 8, *for* (A) and (B) *read* (A) and (C).

VOL. XIII.

p. 41, line 1, *for* (2) becomes *read* (1) becomes,
„ line 8, *for* (2) *read* (1);
p. 112, line 5 from bottom, *for* $\frac{5}{2}r^2 \sin\psi$ *read* $\frac{5}{4}r^2 \sin\psi$.

VOL. XIV.

p. 44, line 2 from bottom, *for* $(a+bx^n)$, &c. *read* $(a+bx^n)^p$, &c.;
p. 45, line 4, *for* $\int x^m$ *read* $\int x^{m-n}$;
„ line 13, *for* $\int x$ *read* $\int x^m$;
p. 48, line 6 from bottom, *for* (3) *read* (2),
„ last line, *for* y' *read* y^2;
p. 95, line 1, *for* $\sin\beta$ *read* $\sin\alpha$;
p. 96, line 16, *for* $\{\ldots\}_2$ *read* $\{\ldots\}^2$.

VOL. XV.

p. 24, line 16, *for* $(2n-1)^3$ *read* $(2n-1)\,x^3$;
p. 36, line 25, *for* another page *read* page 49;
p. 38, top of page, *for* 8 *read* 38;
p. 75, line 3, *for* $\frac{r^2-a^2-x^2}{2ar}$ *read* $\frac{r^2+a^2-x^2}{2ar}$,
„ line 4, *for* $-\frac{1}{2\pi r^2}\{\ \}$ *read* $+\frac{1}{2\pi r^2}\{\ \}$,
„ line 5, *for* $-\frac{a\,(4r^2-a^2)^{\frac{1}{2}}}{2\pi r^2}$ *read* $+\frac{a\,(4r^2-a^2)^{\frac{1}{2}}}{2\pi r^2}$,
„ line 6, *read* $p=\frac{1}{3}+\frac{\sqrt{3}}{2\pi}$;
p. 91, line 1, the bracket after $\frac{\lfloor 5}{\lfloor 3}$ should come in after $\frac{\lfloor 6}{\lfloor 3\,\lfloor 2}$.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

3206. (Proposed by Professor CAYLEY.)—In how many geometrically distinct ways can nine points lie in nine lines, each through three points?

3278. (Proposed by Professor CAYLEY.)—It is required, with nine numbers each taken three times, to form nine triads containing twenty-seven distinct duads (or, what is the same thing, no duad twice), and to find in how many essentially distinct ways this can be done.

Solution by the PROPOSER.

Let the numbers be 1, 2, 3, 4, 5, 6, 7, 8, 9. Any number, say 1, enters into three triads, no two of which have any number in common. We may take these triads to be 123, 145, 167. There remain the two numbers 8, 9; and these are, or are not, a duad of the system.

First Case—8 and 9 a duad. In the triad which contains 89, the remaining number cannot be 1; it must therefore be one of the numbers 2, 3; 4, 5; 6, 7; and it is quite immaterial which; the triad may therefore be taken to be 289. There is one other triad containing 2, the remaining two numbers thereof being taken from the numbers 4, 5; 6, 7. They cannot be 4, 5 or 6, 7; and it is indifferent whether they are taken to be 4, 6; 4, 7; 5, 6, or 5, 7: the triad is taken to be 247. We have thus the triads 123, 145, 167, 289, 247;

and we require two triads containing 8 and two triads containing 9. These must be made up with the numbers 3, 4, 5, 6, 7: but as no one of them can contain 47, it follows that, of the two pairs which contain 8 and 9 respectively, one pair must be made up with 3, 5, 6, 7, and the other pair with 3, 5, 6, 4; say, the pairs which contain 8 are made up with 3, 5, 6, 7, and those which contain 9 are made up with 3, 5, 6, 4 (since obviously no distinct case would arise by the interchange of the numbers 8, 9). The

triads which contain 8 must contain each of the numbers 3, 5, 6, 7, and they cannot be 835, 867 (since we have 67 in the triad 167); similarly the triads which contain 9 must contain each of the numbers 3, 5, 6, 4, and they cannot be 845, 836 (since we have 45 in 145). Hence the triads can only be

836, 857	934, 956;
837, 856	935, 946

and clearly the top row of 8 must combine with the top row of 9, and the bottom row of 8 with the bottom row of 9; that is, the system of the nine triads is 123, 145, 167, 289, 247,

in combination with 836, 857, 934, 956,

or else in combination with 837, 856, 935, 946.

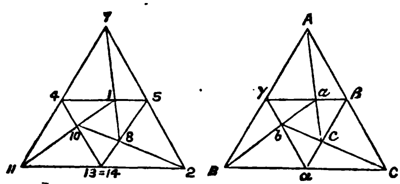
These are really systems of the same form, viz., each of them is of the form

BCα	$\beta\gamma a$	bcC
CAβ	γab	caA
ABγ	$\alpha\beta c$	abB

viz., in the first and second systems respectively we have

A	B	C	α	β	γ	a	b	c	
6	1	3	2	8	7	5	4	9	(First system)
5	1	3	2	9	4	6	7	8	(Second system),

as one out of many ways of effecting the identification. Observe that there is not in the system any triad of triads containing all the numbers. It thus appears that 8, 9, a duad, gives only a single form of the system.



Cor. — It is possible to find in a plane nine points such that the points belonging to the same triad lie *in lineâ*. The nine points are, in fact, on a cubic curve; and the figure is that belonging to a theorem of Prof. Sylvester's, according to which it is possible to find on a cubic curve a system of points 1, 2, 4, 5, 7, 8, &c. (series of numbers not divisible by 3), such that for any triad (such as 145) where the sum of the numbers, one taken negatively, = 0, the three points are *in lineâ*; and so also that, if two of the points become identical, in the figure 13 = 14, then there is not any new point, but the preceding points are indefinitely repeated; thus, 2, 14, 16 being *in lineâ*, and 14 being = 13, 16 must be = 11, and so on.

Second and Third Cases—8 and 9 do not form a duad. There are thus three triads composed of 8 with (2, 3; 4, 5; 6, 7), and three triads composed of 9 with (2, 3; 4, 5; 6, 7). If with these numbers (2, 3; 4, 5; 6, 7) we form all the arrangements of three duads other than those which contain all or any of the duads 23, 45, 67, these are the eight arrangements

A = 24, 37, 56	E = 26, 35, 47
B = 24, 36, 57	F = 26, 34, 57
C = 25, 36, 47	G = 27, 34, 56
D = 25, 37, 46	H = 27, 35, 46

where A has a duad in common with B, with D, and with G, but it has no duad in common with C, E, F, or H. We have thus the sixteen pairs

AC, AE, AF, AH,
BD, BE, BG, BH,
CF, CG, CH,
DE, DF, DG,
EG, FH,

where each pair contains six different duads.

Combining AC with 8, 9, we have the triads 8 (24, 37, 56) and 9 (24, 36, 57), that is, the triads

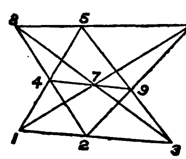
824, 837, 856; 924, 936, 957,

which, with the original three triads 123, 145, 167, form a system of nine triads; 8 and 9 might, of course, be interchanged, but no essentially distinct system would arise thereby. Hence we have a system of nine triads by combining the original three triads 123, 145, 167 with any one of the sixteen pairs AC, AE, &c. But it is sufficient to consider the combinations of the three triads with each of the pairs AC, AE, AF, AH; in fact, these are the only systems which contain the triad 824; and since there is no distinction between the two pairs 4, 5 and 6, 7, or between the two numbers of the same pair, it is allowable to take 824 as a triad of the system. Hence—

Second Case.—The system consists of the three triads combined with AE; viz., it is 123, 145, 167; 824, 837, 856; 926, 935, 947,

which, it is to be observed, consists of three triads of triads, each triad of triads containing all the nine numbers; viz., the system is

123, 479, 568; 145, 269, 378; 167, 248, 359.



Cor.—We may have nine points such that the points belonging to the same triad lie *in lined*, viz., the figure is that of Pascal's hexagon when the conic is a line-pair.

Third Case.—Combining the three triads with AC, AF, or AH, it is readily seen that we obtain in each case a system of the form

$$\begin{array}{lll} A\alpha\alpha', & A\beta\gamma, & A\beta'\gamma' \\ B\beta\beta', & B\gamma\alpha, & B\gamma'\alpha' \\ C\gamma\gamma', & C\alpha\beta, & C\alpha'\beta', \end{array}$$

viz., in the case where the pair is AC; that is, the system is

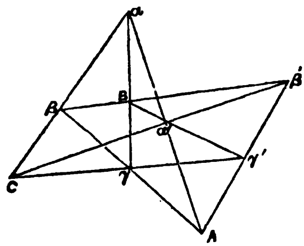
123, 145, 167; 824, 837, 856; 925, 936, 947;

and in the cases where the pair is AF or AH, the identifications may be taken to be

A	B	C	α	β	γ	α'	β'	γ'	
9,	8,	1;	4,	5,	2;	7,	6,	3	.... (AC)
9,	8,	1;	2,	3,	4;	6,	7,	5	.... (AF)
9,	8,	1;	5,	4,	6;	3,	2,	7	.... (AH).

Observe that there is in the system a single triad of triads $A\alpha\alpha'$, $B\beta\beta'$, $C\gamma\gamma'$, containing all the numbers; viz., for the system with AC, this is 123, 856, 947; for the system with AF, it is 145, 837, 926; and for the system with AH, it is 167, 824, 935.

Cor.—It is possible to find a system of nine points such that the points belonging to the same triad lie *in lined.* Such a figure is this:—



The solution shows that these are the only systems of nine points satisfying the prescribed conditions.

PROBABILITY NOTATION. *By* HUGH M'COLL.

Definition:—The symbol $\int \phi(x)\,px$ expresses the limiting value of $\Sigma\phi(x)\,\Delta x$, when Δx is diminished without limit and the number of terms increased without limit, subject to these two conditions:—*That any positive value of $\phi(x)$ greater than unity is to be considered as unity; and that any negative value of $\phi(x)$ is to be taken as zero.* Hence we get the following rules:—

1. If $\phi(x)$ is negative for all values of x between a and b, then

$$\int_a^b \phi(x)\,px = \int_a^b 0\,dx = 0.$$

2. If $\phi(x)$ is positive and less than unity for all values of x between a and b, then
$$\int_a^b \phi(x)\,px = \int_a^b \phi(x)\,dx;$$
that is to say, dx may be substituted for px.

3. If $\phi(x)$ is positive and greater than unity for all values of x between a and b, then
$$\int_a^b \phi(x)\,px = \int_a^b 1\,dx = b-a.$$

As a simple example, take the integral $\int_1^9 (5-x)\,px$. Now the expression $5-x$ is negative for all values of x between 9 and 5; positive and less than unity for all values of x between 5 and 4; and positive and greater

than unity for all values of x between 4 and 1. Hence we get

$$\int_1^9 (5-x)\,px = \int_5^9 0\,dx + \int_4^5 (5-x)\,dx + \int_1^4 1\,dx\,;$$

and this, of course, $= 0 + \int_4^5 (5-x)\,dx + (4-1) = 3\frac{1}{2}$.

4. Let $\phi(x)$ denote, for all *given* values of x between a and b, the probability [subject to the conditions that any positive value of $\phi(x)$ greater than unity is to be considered as unity, and that any negative value of $\phi(x)$ is to be considered as zero] that an event dependent on the value of x will happen. Then the probability that this event will happen when x is taken *at random* between a and b is

$$\int_a^b \phi(x)\,px \div (b-a),$$

the average value of the probability $\phi(x)$ as x passes from a to b.

5. As an extension of the foregoing, let $\phi_1(x)$ denote (with the same restrictions as before) the probability that an event will happen so long as x is *given* between a_1 and a_2; $\phi_2(x)$ the probability that it will happen when x is between a_2 and a_3; $\phi_3(x)$ the probability that it will happen when x is between a_3 and a_4; and so on; a_1, a_2, a_3, &c. being in order of magnitude. Then the probability that it will happen when x is taken *at random* between a_1 and a_n is

$$\left\{\int_{a_1}^{a_2} \phi_1(x)\,px + \int_{a_2}^{a_3} \phi_2(x)\,px + \ldots + \int_{a_{n-1}}^{a_n} \phi_{n-1}(x)\,px\right\} \div (a_n - a_1),$$

the average value of the probability as x passes from a_1 to a_n.

6. To find the probability that any function $\phi(x, y, z, \ldots, u, v)$ will satisfy any condition M, when x, y, z, &c. are all taken at random between their respective limits. First find the probability that M will be satisfied when one alone of the variables, as v, is taken at random, the values of the others being supposed given. Next, find the average value of this probability (the expression or expressions for which no longer involve v) when another of the variables, as u, is taken at random. The expression or expressions for the probability will then contain neither v nor u. Then find the average value of the probability when another of the variables is taken at random; and so on, till all the variables are eliminated, and the probability is expressed in known quantities.

As an illustration, take the following problem:—

"In the quadratic equation $ax^2 - bx + c = 0$, the numerical values of a, b, c are each taken at random between 1 and 10; find the chance of the roots of the equation being real."

The chance of the roots being real is the chance of $b^2 - 4ac$ being positive. First, suppose c alone to be taken at random, a and b being constant. Now $b^2 - 4ac$ will be positive when c is less than $\frac{b^2}{4a}$, and not otherwise; and the chance of this between the assigned limits of c is

evidently $\frac{1}{9}\left(\frac{b^2}{4a}-1\right)$. The average value of this chance as b passes from 1 to 10 is $\frac{1}{9}\int_1^{10}\frac{1}{9}\left(\frac{b^2}{4a}-1\right)pb$ (see Rule 4).

Now when a is less than $\frac{5}{2}$, we get (see Rules 1, 2, 3)

$$\int_1^{10}\frac{1}{9}\left(\frac{b^2}{4a}-1\right)pb=\int_{\sqrt{(40a)}}^{10}1\,db+\int_{\sqrt{(4a)}}^{\sqrt{(40a)}}\frac{1}{9}\left(\frac{b^2}{4a}-1\right)db+\int_1^{\sqrt{(4a)}}0\,db$$

$$=10-\frac{4}{27}\left(10^{\frac{3}{2}}-1\right)a^{\frac{1}{2}},$$

when simplified; so that the chance when b and c are taken at random between 1 and 10, *while a remains constant between* 1 *and* $\frac{5}{2}$, is

$$\frac{10}{9}-\frac{4}{243}\left(10^{\frac{3}{2}}-1\right)a^{\frac{1}{2}}=\phi(a)\ \text{say}.$$

And when a is greater than $\frac{5}{2}$, we get (see Rules 1, 2)

$$\int_1^{10}\frac{1}{9}\left(\frac{b^2}{4a}-1\right)pb=\int_{\sqrt{(4a)}}^{10}\frac{1}{9}\left(\frac{b^2}{4a}-1\right)db+\int_1^{\sqrt{(4a)}}0\,db$$

$$=\frac{2}{27a}\left(125+2a^{\frac{3}{2}}-15a\right);$$

so that the chance when b and c are taken at random between 1 and 10, *while a remains constant between* $\frac{5}{2}$ *and* 10, is

$$\frac{2}{243a}\left(125+2a^{\frac{3}{2}}-15a\right)=\frac{2}{243a}\left(5-a^{\frac{1}{2}}\right)^2\left(5+2a^{\frac{1}{2}}\right)=\phi_1(a)\ \text{say}.$$

If now a be taken at random between 1 and 10, the required chance (see Rule 5) will be $\frac{1}{9}\left\{\int_1^{\frac{5}{2}}\phi(a)\,pa+\int_{\frac{5}{2}}^{10}\phi_1(a)\,pa\right\}$;

and as da may be substituted for pa in both these integrals (see Rule 2), we finally get the required chance to be

$$\frac{1500\log_e 2+160\sqrt{10}-468}{6561}=\cdot 1642\ \text{nearly}.$$

[For another example, see the solution of the next Question, 3279.]

3279. (Proposed by Dr. SYLVESTER.)—If a quadratic equation with real coefficients be written down at hazard, find the probability of its roots being imaginary.

I. *Solution by* Hugh M'Coll.

Let the quadratic equation be $ax^2+bx+c=0$, all values of a, b, c being equally probable. If the roots are imaginary, two independent conditions, and two only, must be satisfied: *first*, a and c must have like signs; and *secondly*, b must be *numerically* less than $2a^{\frac{1}{2}}c^{\frac{1}{2}}$. The probability of the first condition is evidently $\frac{1}{2}$; so that if we denote the probability of the second condition by p, the required probability of imaginary roots will be $\frac{1}{2}p$. It only remains, therefore, to find p, taking into account the *positive values only* of a, b, c.

First, let a, b, c be all restricted between the limits 0 and some finite quantity n; and let us begin by supposing that b alone is taken at random, a and c being constant. In this case it is clear that $p=\dfrac{2a^{\frac{1}{2}}c^{\frac{1}{2}}}{n}$. The average value of this chance as c passes from 0 to n, while a remains constant, is $\dfrac{1}{n}\displaystyle\int_0^n \frac{2a^{\frac{1}{2}}c^{\frac{1}{2}}}{n}\,pc$ (see my article on Probability Notation, Rule 4).

Now when a is greater than $\frac{1}{4}n$, we get (see Rules 2, 3)

$$\frac{1}{n}\int_0^n \frac{2a^{\frac{1}{2}}c^{\frac{1}{2}}}{n}\,pc=\frac{1}{n}\left\{\int_{\frac{n^2}{4a}}^n 1\,dc+\int_0^{\frac{n^2}{4a}}\frac{2a^{\frac{1}{2}}c^{\frac{1}{2}}}{n}\,dc\right\}=1-\frac{n}{12a};$$

and when a is less than $\frac{1}{4}n$, we get (see Rule 2)

$$\frac{1}{n}\int_0^n \frac{2a^{\frac{1}{2}}c^{\frac{1}{2}}}{n}\,pc=\frac{1}{n}\int_0^n \frac{2a^{\frac{1}{2}}c^{\frac{1}{2}}}{n}\,dc=\frac{4}{3}\left(\frac{a}{n}\right)^{\frac{1}{2}}.$$

Hence $$p=\frac{1}{n}\left\{\int_{\frac{1}{4}n}^n\left(1-\frac{n}{12a}\right)pa+\int_0^{\frac{1}{4}n}\frac{4}{3}\left(\frac{a}{n}\right)^{\frac{1}{2}}pa\right\}\quad\text{(see Rule 5).}$$

And since in both these integrals da may be substituted for pa (see Rule 2), we finally get $p=\frac{31}{36}-\frac{1}{6}\log_e 2$, in which n disappears; and therefore $\frac{1}{2}p=\frac{31}{72}-\frac{1}{12}\log_e 2$, which is the chance required, for, being independent of the value of n, it is true when n is infinite. This probability of imaginary roots expressed to 7 places of decimals is $\cdot 3727933$.

II. *Solution by* Stephen Watson.

Let u, v, w be the coefficients; then, understanding the question to be, that each value between $\pm\infty$, of each of the quantities u, v, w, is equally likely to be written down, we have for imaginary roots $v^2<4uw$ ...... (1); hence the limits are w from $\dfrac{v^2}{4u}$ to a, u from $\dfrac{v^2}{4a}$ to a, and v from 0 to a, where a is the limit within which we suppose u, v, w, for the present, to lie.

Now (1) is always impossible when either u or w is negative; but when u and w are both negative, it remains the same, therefore we must multiply by 2, and then again by 2, because v may be either positive or negative. Hence, dividing by $8a^3$ the number of values u, v, w can take, we have

$$\frac{4}{8a^3}\int_0^a dv \int_{\frac{v^2}{4a}}^a du \int_{\frac{v^2}{4u}}^a dw = \frac{4}{8a^3}\int_0^a dv \left(a^2 - \tfrac{1}{4}v^2 + \tfrac{1}{4}v^2 \log \frac{v^2}{4a^2}\right)$$

$$= \tfrac{31}{72} - \tfrac{1}{12}\log 2,$$

the required chance, since it holds good when $a = \pm\infty$.

For real roots the chance is $= \frac{41}{72} + \frac{1}{12}\log 2$, or ·6272 nearly.

3158. (Proposed by Dr. JAMES MATTESON.)—To find a cube number of numbers which are cubes whose roots are consecutive numbers in the natural series.

I. *Solution by* Dr. DAVID S. HART.

Let $\{x-(n-1)\}^3$, $\{x-(n-2)\}^3$, $\{x-(n-3)\}^3 \ldots x^3 \ldots \{x+(n-3)\}^3$, $\{x+(n-2)\}^3$, $\{x+(n-1)\}^3$, be an *odd* series of cube numbers whose roots are consecutive numbers in the natural series; x^3 being the middle term.

Then, beginning at this term, the sum of 1, 3, 5, 7, 9, 11, 13, ... terms is

$$x^3;\quad (x-1)^3+x^3+(x+1)^3,\ \text{ or }\ 3x^3+6x;$$

$$(x-2)^3+(x-1)^3+x^3+(x+1)^3+(x+2)^3,\ \text{ or }\ 5x^3+30x;$$

$$7x^3+84x;\quad 9x^3+180x;\quad 11x^3+330x;\quad 13x^3+546x;\quad \&c.$$

Therefore the sum of $2n-1$ terms is $= (2n-1)^3+(2n^3-3n^2+n)\,x \ldots\ldots(1)$;

for the nth term of the series 1, 3, 5, 7, 9, 11, 13, &c., is $2n-1$, and the $(n-1)$th term of the series 6, 30, 84, 180, 330, 546, &c., is $2n^3-3n^2+n$. But the problem requires that (1) shall be a cube, and also that $2n-1$ shall be a cube.

Let $2n-1 = p^3$; then $n = \frac{1}{2}(p^3+1)$, and, by substitution, (1) becomes $p^3x^3 + \frac{1}{4}p^3(p^6-1)\,x$ = a cube; or, dividing by p^3, $x^3+\frac{1}{4}(p^6-1)\,x$ = a cube. This expression will be a cube when $x=\frac{1}{2}$. Let $x = y+\frac{1}{2}$; then, by substitution, multiplying by 8, and arranging the terms, we have

$$8y^3+12y^2+(2p^6+4)\,y+p^6 = \text{a cube, which put} = (2y+p^2)^3;\ \text{ whence}$$

$$y = \frac{p^6-3p^4+2}{6\,(p^2-1)} = \frac{(p^2-1)^2-3}{6},\ \text{ and }\ x = \frac{(p^2-1)^2}{6} = \frac{(p-1)^2\,(p+1)^2}{6}.$$

Here the value of x must be integral to have consecutive numbers. This will be effected by taking $p = 6m \pm 1$, m being any number.

Let $m = 1$; then, using the negative sign, $p = 5$, $x = 96$, and $n = 63$. Substituting these values in the original series, we have 34, 35, 36, ... 156, 157, 158 for the *roots* of 125 consecutive cubes in the natural series of numbers.

Using the plus sign, $p=7$; then $x=384$, $n=172$, and, by substitution, we have 213 ... 555 for the roots of 343 cubes which fulfil the conditions of the problem.

If $m=2$, using the minus sign, we have 1331 cubes; and taking the plus sign $p=13$, we shall have 2197 cubes.

Now to find an even number of cubes, we have only to add to formula (1) the term in the original series next to $\left\{x+(n-1)\right\}^3$, viz., $(x+n)^3$, and then we shall have

$$(2n-1)x^3+(2n^3-3n^2+n)x+(x+n)^3=2nx^3+3nx^2+(2n^3+n)x+n^3=\text{a cube}.$$

But $2n$ must also be a cube. Let $2n=p^3$; then $n=\frac{1}{2}p^3$, and, by substitution, we have $p^3x^3+\frac{3}{2}p^3x^2+\frac{1}{4}(p^9+2p^3)x+\frac{1}{8}p^9=$ a cube, which, after dividing by p^3 and multiplying by 8, becomes

$$8x^3+12x^2+(2p^6+4)x+p^6=\text{a cube, which put }=(2x+p^2)^3.$$

Reducing, we find

$$x=\frac{p^6-3p^4+2}{6(p^2-1)}=\frac{(p^2-1)^2-3}{6}=\frac{(p-1)^2(p+1)^2-3}{6};$$

where, in order to have x integral, p may be taken any even number, except 6, and its multiples. Let $p=2$; then $x=1$, $n=4$. Substituting these values in the original series, we have $-2, -1, 0, 1, 2, 3, 4, 5$, four of which numbers balance one another, and one is 0; whence we obtain 3, 4, 5, the roots of three cubes whose sum is 6^3.

Let $p=4$; then $x=37$, $n=32$, and by substitution we find 6, 7, 8, 9, ... 66, 67, 68, 69 for the roots of 64 cubes that fulfil the conditions of the problem.

Let $p=8$; then $x=661$, $n=256$, and we have the series 406, 407, ... 916, 917 for the roots of 512 consecutive cubes in the natural series of numbers.

Let $p=10$; then $x=1633$, $n=500$, and by substitution we have 1134, 1135, 1136, ... 2131, 2132, 2133 for the roots of 1000 cubes that fulfil the conditions.

II. *Solution by* Asher B. Evans, M.A.

Let $(x+1)$ be the first of the n^3 consecutive numbers; then, since

$$1^3+2^3+3^3+\ldots+x^3=\left\{\tfrac{1}{2}x(x+1)\right\}^2,$$

and
$$1^3+2^3+3^3+\ldots+(x+n^3)^3=\left\{\tfrac{1}{2}(x+n^3)(x+n^3+1)\right\}^2;$$

therefore
$$\left\{\tfrac{1}{2}(x+n^3)(x+n^3+1)\right\}^2-\left\{\tfrac{1}{2}x(x+1)\right\}^2$$
$$=n^3\left\{x^3+\tfrac{3}{2}(n^3+1)x^2+\tfrac{1}{2}(2n^6+3n^3+1)x+\tfrac{1}{4}n^3(n^3+1)^2\right\}$$

must be equal to a cube. Let $nx+\frac{1}{2}(n^2+n^4)$ be the root of this cube; then

$$8x^3+12(n^3+1)x^2+4(2n^6+3n^3+1)x+2n^3(n^3+1)^2=(2x+n^2+n^3)^3.$$

The preceding equation readily reduces to the quadratic

$$12(n^2-1)x^2-2(n^2-1)(n^4-6n^3-2n^2-2)x=n^3(n^2-1)(n^4-3n^3-2n^2-2).$$

By dividing this equation by (n^2-1), and solving the result with reference to x, we obtain
$$x=\tfrac{1}{6}(n^4-3n^3-2n^2-2),$$
where n may have any integral value greater than three.

[By giving n the values 5, 7, 4, 8, 10, Mr. Evans's formula gives the same results as those which have been obtained above by Dr. Hart.]

III. *Solution by* A. MARTIN; *and* A. MCLEAN.

Let x, $(x+1)$, $(x+2)$, ... $(x+n^3-1)$ be the numbers; the sum of their cubes is $\left\{\frac{1}{2}(x+n^3-1)(x+n^3)\right\}^2-\left\{\frac{1}{2}x(x-1)\right\}^2$, $=a^3$, suppose; then

$$\left\{2x^2+2(n^3-1)x+(n^3-1)n^3\right\}(2x+n^3-1)n^3 = 4a^3 = 4b^3n^3, \text{ suppose};$$

$$\therefore \quad 8x^3+12(n^3-1)x^2+4(n^3-1)(2n^3-1)x+2n^3(n^3-1)^2 = 8b^3 = \text{a cube}.$$

Assume $2b = 2x+(n-1)(n+1)^2+(n-1)$, then we have

$$12x^2-2(n^4-6n^3-2n^2+10)x = n^7-3n^6-2n^5-2n^4+10n^3+4n^2-8.$$

Multiplying by 12, and then adding $(n^4-6n^3-2n^2+10)^2$ to both sides, we have $\left\{12x-(n^4-6n^3-2n^2+10)\right\}^2 = n^8-4n^6+8n^2+4 = (n^4-2n^2-2)^2$;

$\therefore \quad 12x-(n^4-6n^3-2n^2+10) = n^4-2n^2-2$, and $x = \frac{1}{6}(n^4-3n^3-2n^2+4)$.

Taking $n=4$, we have $x=6$; thus 6, 7, 8, 9, ... 69 are 64 consecutive numbers, the sum of whose cubes is a cube $= 180^3$.

3262. (Proposed by C. W. MERRIFIELD, F.R.S.)—Let a pencil of five lines in space be met by a transversal plane. Find ratios which are independent of the position of the plane.

Solution by the PROPOSER.

Let the centre of the pencil be O, and consider any triangle ABC. Also let the perpendicular from O on the plane be p. Then the volume of the pyramid from O on ABC may be expressed in two ways, namely,

$$(1) \ldots\ldots \tfrac{1}{3}p\,.\,\text{ABC}, \qquad (2) \ldots\ldots \text{OA}\,.\,\text{OB}\,.\,\text{OC}\,.\,\phi\,(\text{ABC}),$$

where $\phi\,(\text{ABC})$ is a function only of the angles AOB, BOC, COA. If, therefore, we can express any combination of the areas of the triangles which can be formed by five points, simply by means of the quantities $\phi\,(\text{ABC})$, &c., which only depend on the angles, we shall attain our object.

Take for instance the expression $\dfrac{\text{ABC}\,.\,\text{ADE}}{\text{ABD}\,.\,\text{ACE}}$. Multiply both numerator and denominator by $\frac{1}{9}p^2$, and substitute for $\frac{1}{3}p\,.\,\text{ABC}$ the product $\text{OA}\,.\,\text{OB}\,.\,\text{OC}\,.\,\phi\,(\text{ABC})$, and so on. The result is

$$\frac{\text{OA}\,.\,\text{OB}\,.\,\text{OC}\,.\,\text{OA}\,.\,\text{OD}\,.\,\text{OE}}{\text{OA}\,.\,\text{OB}\,.\,\text{OD}\,.\,\text{OA}\,.\,\text{OC}\,.\,\text{OE}} \times \frac{\phi\,(\text{ABC})\,.\,\phi\,(\text{ADE})}{\phi\,(\text{ABD})\,.\,\phi\,(\text{ACE})}.$$

The first fraction is evidently unity. The ratio $\dfrac{\text{ABC}\,.\,\text{ADE}}{\text{ABD}\,.\,\text{ACE}}$ is therefore one of those which answer the question.

$\phi\,(\text{ABC})$ is simply the well known angular function which enters into the expression for the volume of a pyramid whose rising edges and the angles between them are given.

It remains to find what are the different combinations which will answer the purpose.

The combinations 3 together of 5 numbers are 10, and these can be divided into 5 and 5 in 126 different ways. But among these are only 6 ways in which the letters A, B, C, D, E will enter symmetrically; that is to say, such that each shall appear three times in each combination of five triangles. These combinations are

(1) ABC . ABE . ADE . BCD . CDE, (7) ABD . ACD . ACE . BCE . BDE,
(2) ABC . ABE . ACD . BDE . CDE, (8) ABD . ACE . ADE . BCD . BCE,
(3) ABC . ACE . ADE . BCD . BDE, (9) ABD . ABE . ACD . BCE . CDE,
(4) ABC . ACD . ADE . BCE . BDE, (10) ABD . ABE . ACE . BCD . CDE,
(5) ABC . ABD . ACE . BDE . CDE, (11) ABE . ACD . ADE . BCD . BCE,
(6) ABC . ABD . ADE . BCE . CDE, (12) ABE . ACD . ACE . BCD . BDE.

If we divide any one of these 12 quantities by any other, we shall have a ratio which remains constant for a variable transversal plane.

The ratios $\frac{(1)}{(7)}, \frac{(2)}{(8)}, \frac{(3)}{(9)}, \frac{(4)}{(10)}, \frac{(5)}{(11)}, \frac{(6)}{(12)}$, and their reciprocals, have no factors which divide out. But if we pair them in any other way, we lose either two or three factors. We get in this way the following types:—

$$\text{(i.)}\ \frac{ABC \cdot ADE}{ABD \cdot ACE}, \qquad \text{(iv.)}\ \frac{ABE \cdot BCD \cdot CDE}{ACD \cdot BCE \cdot BDE},$$
$$\text{(ii.)}\ \frac{ABD \cdot ACE}{ABE \cdot ACD}, \qquad \text{(v.)}\ \frac{ABC \cdot BDE \cdot CDE}{ADE \cdot BCD \cdot BCE},$$
$$\text{(iii.)}\ \frac{ABE \cdot ACD}{ABC \cdot ADE}, \qquad \text{(vi.)}\ \frac{ABD \cdot BCE \cdot CDE}{ACE \cdot BCD \cdot BDE}.$$

By introducing the cyclic change, we see that there are (including reciprocals) 30 binary ratios, and 30 ternary ratios, and we can at once write them down. The five-term ratios may be got from any group of these. Thus we have

$$\text{(i.)} \times \text{(iv.)} = (1) \div (7), \qquad \text{(i.)} \div \text{(iv.)} = (4) \div (10),$$
$$\text{(v.)} \times \text{(ii.)} = (5) \div (11), \qquad \text{(v.)} \div \text{(ii.)} = (2) \div (8),$$
$$\text{(vi.)} \times \text{(iii.)} = (9) \div (3), \qquad \text{(vi.)} \div \text{(iii.)} = (6) \div (12).$$

Neglecting reciprocals, there are thus 15 different binary ratios, and 15 ternary ratios, which fulfil the conditions of the question. All others are functions of these—at least, all others made up of triangular areas, or, what is equivalent, the volume-ratios of pencils of three lines.

A convention is requisite as to sign. Probably that used in determinants would serve the purpose.

As a particular case, let BCED be a quadrilateral, of which the diagonals BE and CD meet in A. Then we evidently have

$$\frac{ABC \cdot ADE}{ABD \cdot ACE} = -1.$$

In the corresponding ternary combination, (iv.) $\frac{ABE \cdot BCD \cdot CDE}{ACD \cdot BCE \cdot BDE}$, ACD and ABE vanish. But nevertheless it is easy to show that $\frac{BCD \cdot CDE}{BCE \cdot BDE} = 1$.

The quadrilateral seems to give the simplest case of the ratio. The property is from its nature projective, and is obviously true of all quadrilaterals.

3044. (Proposed by S. ROBERTS, M.A.)—Planes passing through an axis of an ellipsoid cut it in sections whose real foci lie in the plane of the other axes. Find the locus of the foci which is of the fourth degree.

Solution by the PROPOSER.

Let a, b, c, written in the order of decreasing magnitude, be the axes of the ellipsoid whose equation is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

Turning the axes of x and y about the axis of z though the same angle θ we may write the equation

$$\frac{(x'\cos\theta-y'\sin\theta)^2}{a^2}+\frac{(x'\sin\theta+y'\cos\theta)^2}{b^2}+\frac{z^2}{c^2}=1.$$

If $y'=0$, we have $$\frac{x'^2}{\dfrac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}}+\frac{z^2}{c^2}=1;$$

therefore $$\rho^2=\frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}-c^2$$

is the polar equation of the locus, or

$$(x^2+y^2)(a^2y^2+b^2x^2)-b^2(a^2-c^2)x^2-a^2(b^2-c^2)y^2=0.$$

3122. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—On the normal at any point P of an ellipse is measured inwards PO equal to half the conjugate diameter, and from O are drawn three other normals, OL, OM, ON to the ellipse: prove that the triangle LMN will bear to the triangle formed by the tangents at L, M, N a ratio independent of the position of P.

Solution by J. J. WALKER, M.A.

More generally, if O be so assumed that PO may bear any constant ratio (k) to the conjugate semi-diameter, the ratio of the triangles in question will be independent of the position of P. The ellipse being $b^2x^2+a^2y^2-a^2b^2=0$, let P be (x', y'), O (x, y); then, if $a-kb=m$, $b-ka=n$, it is easily shown that $x=\dfrac{m}{a}x'$, $y=\dfrac{n}{b}y'$ [consequently the locus of O is $\dfrac{x^2}{m^2}+\dfrac{y^2}{n^2}-1=0$], and it is well known that x' is a root of

$$c^4x'^4-2a^2c^2xx'^3+a^2(a^2x^2+b^2y^2-c^4)x'^2+2a^4c^2xx'-a^6x^2=0.$$

Dividing this by $x'-\dfrac{a}{m}x$, there results

$$c^4mx'^3-ac^2(am+bn)xx'^2+a^3m^2(2bn-am)x'+a^5m^2x=0$$

to determine the x's of L, M, N. Let these points be (x_1, y_1), (x_2, y_2), (x_3, y_3), and let the tangents at MN meet in L′... N′; then it is easily found that

$$\frac{\text{area LMN}}{\text{area L'M'N'}} = \frac{(x_2y_3-x_3y_2)(x_3y_1-x_1y_3)(x_1y_2-x_2y_1)}{a^2b^2(x_2y_3-x_3y_2+x_3y_1-x_1y_3+x_1y_2-x_2y_1)}.$$

But $y_1 = \dfrac{b^2yx_1}{a^2x-c^2x_1}$, $y_2 = \ldots\ldots$; and substituting these values in the preceding fraction, it reduces to

$$\frac{b^2c^4y^2x_1^2x_2^2x_3^2}{a^4x(a^2x-c^2x_1)(a^2x-c^2x_2)(a^2x-c^2x_3)},$$

which, since x_1, x_2, x_3 are roots of the cubic in x' above, is equal to

$$\frac{b^2c^4y^2a^{10}m^4x^2}{a^4c^4mx\left\{a^6c^4mx^3-a^5c^4(am+bn)x^3+a^5c^4m^2(2bn-am)x+a^5c^6x\right\}}.$$

This reduces identically (since $2bn-am+c^2=bn$) to

$$\frac{abm^3y^2}{c^4n(m^2-x^2)} \text{ or } \frac{abmn}{c^4},$$

in virtue of $m^2y^2=n^2(m^2-x^2)$. Hence

$$\frac{\text{LMN}}{\text{L'M'N'}} = \frac{ab(a-kb)(b-ka)}{c^4}.$$

2833. (Proposed by Patrick O'Cavanagh.)—1. Through every point A on a central conic pass three circles, which osculate the circle elsewhere, say in B, C, D. Prove that the diameter of the circle passing through ABCD is $= \dfrac{a^2+3b^2}{2b}$ when greatest, and $= \dfrac{b^2+3a^2}{2a}$ when least; and when this circle is given in magnitude, show how to find (by rule and compass) its position with respect to the conic.

2. Prove also that the distance between the centre of circle through ABCD and the centre of the circle osculating the conic at B, (C or D,) is when greatest $= \dfrac{3}{4}\dfrac{a^2-b^2}{b}$, and when least $= \dfrac{3}{4}\dfrac{a^2-b^2}{a}$, where a and b are, as usual, the semi-axes of the conic.

Solution by J. J. Walker, M.A.

1. The equation of the circle ABCD (x', y' being coordinates of A) is

$$2(x^2+y^2)-(a^2-b^2)\left(\frac{x'x}{a^2}-\frac{y'y}{b^2}\right)-(a^2+b^2)=0 \;\ldots\ldots\ldots\ldots (1);$$

hence, if (X, Y) be centre and R radius, b' a semi-diameter parallel to the tangent at A,

$$X = \frac{a^2-b^2}{4a^2}x', \quad Y = \frac{b^2-a^2}{4b^2}y', \quad 4R^2 = \frac{(a^2-b^2)^2}{4a^2b^2}b'^2+2(a^2+b^2);$$

so that $4R^2$ will be a maximum when $b' = a$, a minimum when $b' = b$; and the substitution of these values for b' gives the values of $2R$ announced in the question. [If R be given, b' and therefore A are given, and the centre of the circle is readily constructed with the aid of the last property given in the solution of Question 2852.]

2. Let (x_1, y_1) be one of the three points B, C, D, and α, β the coordinates of centre of osculating circle at that point; then x_1, y_1 are connected with x', y' by the relations

$$a^2x' = 4x_1^3 - 3a^2x_1, \quad b^2y' = 4y_1^3 - 3b^2y_1,$$

also $$\alpha = \frac{a^2-b^2}{a^4}x_1^3, \qquad \beta = \frac{b^2-a^2}{b^4}y_1^3;$$

whence, and from the values of X, Y given above,

$$\alpha - X = \frac{3}{4}\frac{a^2-b^2}{a^2}x_1, \quad \beta - Y = \frac{3}{4}\frac{b^2-a^2}{b^2}y_1,$$

and $$\left\{(\alpha-X)^2+(\beta-Y)^2\right\}^{\frac{1}{2}} = \frac{3}{4}\frac{a^2-b^2}{ab}b_1$$

[where b_1 is the semi-diameter parallel to the tangent at (x_1, y_1)], which will evidently be a maximum when $b_1 = a$, a minimum when $b_1 = b$; so that the maximum and minimum values are those given in the question.

2319. (Proposed by R. TUCKER, M.A.)—If O, O′ be the centres of two circles, prove that the polar of any point on (O) with reference to (O′) envelopes a conic of which O′ is a focus.

Solution by the PROPOSER.

Refer the two circles to rectangular axes, through O′O and perpendicular to it; then the equations to the circles may be written

$$x^2+y^2 = a^2, \quad x^2+y^2+2bx-c^2 = 0 \;\ldots\ldots\ldots\ldots\ldots (1), (2).$$

The equation to the polar of a point (x', y') on (2) is $xx'+yy' = a^2$; and for the determination of the envelope we have to eliminate (x', y') between $xx'+yy' = a^2$, $x'y - xy' = -by$, and (2); this result is

$$c^2\left(x - \frac{a^2b}{c^2}\right)^2 + (c^2+b^2)\,y^2 = \frac{a^4}{c^2}(b^2+c^2),$$

which is the equation to an ellipse with O′ as focus.

3107. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Prove the following analogue to a well-known property in geometry of two dimensions. If A, B, C, D be the four centres of any four spheres having a common

radical axis, a, b, c, d their four radii, and $\alpha, \beta, \gamma, \delta$ their four angles of intersection with any arbitrary sphere; prove the general relation

$$(BCD)\, a \cos \alpha - (CDA)\, b \cos \beta + (DAC)\, c \cos \gamma - (ABC)\, d \cos \delta = 0,$$

where (BCD) &c. are the areas, regard being had to their signs, as well as to their magnitudes, of the four triangles they severally represent.

I. *Solution by the* Proposer.

For, O being the centre and r the radius of the arbitrary sphere, since

$$OA^2 = a^2 + r^2 - 2ar \cos \alpha, \quad OB^2 = b^2 + r^2 - 2br \cos \beta,$$
$$OC^2 = c^2 + r^2 - 2cr \cos \gamma, \quad OD^2 = d^2 + r^2 - 2dr \cos \delta;$$

therefore at once

$$(BCD)\, OA^2 - \&c. = (BCD)\, a^2 - \&c. + r^2 \{(BCD) - \&c.\}$$
$$- 2r \{(BCD)\, a \cos \alpha - \&c.\};$$

since, again, $(BCD) - \&c. = 0$; and since, finally, from the coaxality of the four spheres,

$$(BCD)\, OA^2 - \&c. = (BCD)\, a^2 - \&c. = (ABCD)\, (AI \,.\, CI - BI \,.\, DI),$$

where I is the intersection of the lines AC and BD; therefore, &c.

Note.—As for the corresponding property in geometry of two dimensions, it follows at once from the above, that a variable sphere, intersecting any three fixed spheres at any three constant angles, intersects at constant angles all spheres coaxal with the three, touches in every position an infinite number of such spheres, and intersects at right angles an infinite number of them also. An obvious solution is also supplied by it of the general problem, "To construct a sphere intersecting any four given spheres at any four given angles."

II. *Solution by* J. J. Walker, M.A.

Let A, B, C, D be four points in a plane; O any arbitrary fifth point; and $x_1, \ldots x_4$ the projections of OA, OB, OC, OD on any axis, in the same plane, through O;

then $\qquad (BCD) - (CDA) + (DAB) - (ABC) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (1),$

and $\qquad x_1 (BCD) - x_2 (CDA) + x_3 (DAB) - x_4 (ABC) = 0 \;\ldots\ldots\ldots\ldots (2).$

For if $y_1 \ldots y_4$ be the projections of OA, OB, OC, OD on an axis through O perpendicular to the first, we shall have

$$2\,(BCD) = x_2 (y_3 - y_4) + x_3 (y_4 - y_2) + x_4 (y_2 - y_3),$$
$$2\,(CDA) = x_3 (y_4 - y_1) + x_4 (y_1 - y_3) + x_1 (y_3 - y_4),$$
$$2\,(DAB) = x_4 (y_1 - y_2) + x_1 (y_2 - y_4) + x_2 (y_4 - y_1),$$
$$2\,(ABC) = x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2);$$

whence it easily appears that the values of (1) and (2) vanish identically.

Now, if four spheres have a common radical axis, the four centres A, B, C, D must lie in the same plane, to which this axis is perpendicular. Let K be the centre and r the radius of the fifth, arbitrary sphere; Q the pro-

jection of K on the plane ABCD, P the point in which the radical axis meets the same plane, and O the middle point of PQ; then

$$2ra\cos\alpha = r^2 + a^2 - \mathrm{KA}^2 = r^2 + a^2 - \mathrm{QA}^2 - \mathrm{KQ}^2 = r^2 - \mathrm{KQ}^2 - t^2 + \mathrm{PA}^2 - \mathrm{QA}^2,$$

if t be the length of the tangent from P to any one of the four given spheres. Let x_1 be the projection of OA on PQ $(=p)$, and write k^2 for $r^2 - \mathrm{KQ}^2 - t^2$. Then $\quad 2ra\cos\alpha = k^2 + 2px_1$;

similarly $\quad 2rb\cos\beta = k^2 + 2rx_2 \ldots\ldots$

Hence we have

$$2r\left\{(\mathrm{BCD})\,a\cos\alpha - (\mathrm{CDA})\,b\cos\beta + (\mathrm{DAB})\,c\cos\gamma - (\mathrm{ABC})\,d\cos\delta\right\}$$
$$= k^2\left\{(\mathrm{BCD}) - (\mathrm{CDA}) + (\mathrm{DAB}) - (\mathrm{ABC})\right\}$$
$$+ 2p\left\{x_1(\mathrm{BCD}) - \ldots - x_4(\mathrm{ABC})\right\} = 0,$$

in virtue of (1), (2).

3239. (Proposed by J. J. WALKER, M.A.)—If u, v be two binary quadrics, A, A′, B the invariants of the system, and w their Jacobian, (1) prove that $\left\{\mathrm{B} \pm 2(\mathrm{AA}')^{\frac{1}{2}}\right\} uv \sim w^2$ is a perfect square; and (2) hence show how to transform u, v into $(a, b, c \between xy)^2$, $(\lambda a, -\lambda b, \lambda c \between xy)^2$ respectively; also, as an example, so transform

$$2x^2 + 6xy + 4y^2 \quad \text{and} \quad 28x^2 - 18xy + 2y^2.$$

Solution by the PROPOSER.

For the forms $ax^2 + 2bxy + cy^2$, $\lambda ax^2 - 2\lambda bxy + \lambda cy^2$ of u, v,

$\mathrm{B} = 2\lambda(ac + b^2)$, $\mathrm{A}' = \lambda^2\mathrm{A}$, $w = 2\lambda b(ax^2 - cy^2)$, $uv = \lambda\left\{(ax^2 - cy^2)^2 + 4\mathrm{A}x^2y^2\right\}$;

whence $\quad 4\lambda b^2 uv - w^2 = 4\lambda\mathrm{A}x^2y^2$ .... (1),

which is a perfect square. But

$\lambda(ac - b^2) = \pm(\mathrm{AA}')^{\frac{1}{2}}$, $\;2\lambda(ac - b^2) = \mathrm{B}$, whence $4\lambda b^2 = -\mathrm{B} \pm 2(\mathrm{AA}')^{\frac{1}{2}}$.

Consequently, and from (1), $\left\{\mathrm{B} \pm 2(\mathrm{AA}')^{\frac{1}{2}}\right\} uv \sim w^2 = 4(\mathrm{AA}')^{\frac{1}{2}} x^2y^2$.

Hence it appears that to transform u, v into the above forms, we should take for the new x and y the factors of the square root of

$$\left\{\mathrm{B} \pm 2(\mathrm{AA}')^{\frac{1}{2}}\right\} uv \sim w^2.$$

In the example $u \equiv 2\mathrm{X}^2 + 6\mathrm{XY} + 4\mathrm{Y}^2$, $v \equiv 28x^2 - 18xy + 2y^2$, it will be found that $w \equiv 102x^2 + 108xy - 42y^2$, $\mathrm{A} = -1$, $\mathrm{A}' = -25$, $\mathrm{B} = 170$, and $\mathrm{B} \pm 2(\mathrm{AA}')^{\frac{1}{2}} = 180$ or 160. Taking 180 first,

$$180uv \sim w^2 = 36\,(9x^4 - 48x^3y + 46x^2y^2 + 48xy^3 + 9y^4),$$

the square root of which is $6\,(3x^2 - 8xy - 3y^2)$, or $6\,(3x + y)(x - 3y)$.

Taking, then, $\mathrm{X} = 3x + y$, $\mathrm{Y} = x - 3y$, or $10x = 3\mathrm{X} + \mathrm{Y}$, $10y = \mathrm{X} - 3\mathrm{Y}$,

u and v are transformed into $2X^2-3XY+Y^2$ and $2X^2+3XY+Y^2$ respectively, to a numerical factor. Similarly, taking the other value of $B\pm 2(AA')^{\frac{1}{2}}$, u, v are transformed into

$$3(12+17\sqrt{-2})X^2 \mp 152XY + 3(12-17\sqrt{-2})Y^2$$

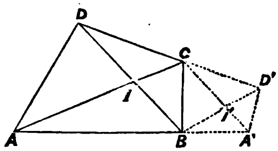
respectively, to numerical factors.

3306. (Proposed by Professor CROFTON.)—1. A frame consists of four joints A, B, C, D, and six rods in a state of tension, uniting the joints in every possible way, no external forces acting. Draw CA′ parallel to DB, meeting AB produced in A′; draw BD′ parallel to AC, meeting DC produced in D′. Prove that the tensions of the rods are represented by the sides and diagonals of the quadrilateral A′BCD′.

If ABCD be inscribable in a circle, it will serve as a diagram of forces for itself, each side representing the tension of the opposite one, each diagonal that of the other.

2. If four forces acting along the sides of a quadrilateral ABCD are in equilibrium, they are proportional to the sides of A′BCD′.

I. *Solution by* PROFESSOR CREMONA.



Les quadrangles complets ABCD, A′BCD′ ont cinq cotés de l'un parallèles respectivement à cinq cotés de l'autre; donc (STANDT, *Geometrie der Lage*, 91) leurs sixièmes cotés AD, A′D′ seront aussi parallèles. D'où il suit que les deux quadrangles sont deux *figures réciproques*, selon la définition de Mr. CLERK MAXWELL (*Philosophical Magazine*, avril, 1864, p. 251), et que par conséquent (*ib.* p. 258), s'il y a équilibre entre des forces agissant suivant les cotés d'un quadrangle, les grandeurs des forces sont données par les cotés correspondants de l'autre quadrangle. On trouve ce même théorème à p. 365 de l'excellente *Graphische Statik* de Mr. CULMANN.

Si A, B, C, D sont quatre points d'un cercle, on a angle ABD = ACD = BD′C, et angle ADB = ACB = CBD′; donc les triangles ABD, CD′B sont semblables. De même pour les autres triangles; d'où il suit que ABCD et CD′A′B sont deux figures semblables. Par conséquent, les tensions des barres BC, CA, AB, AD, BD, CD, qui sont représentées par les cotés BC, BD′, A′B, A′D′, A′C, CD′, sont aussi proportionnelles aux cotés AD, BD, CD, BC, CA, AB.

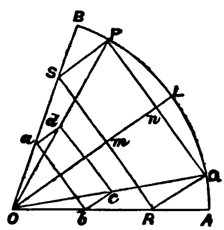
II. *Solution by* Rev. R. TOWNSEND, M.A., F.R.S.

The different parts of this pretty property are evidently immediate consequences from the parallelism of the two lines AD and A′D′, which appears at once from the similarity of the two triangles AID and A′I′D′,

where I and I′ are the intersections of CA and BD, and of CA′ and BD′ respectively, that is of the diagonals of the original co-derived quadrilaterals ABCD and A′BCD′ respectively.

3259. (Proposed by the Editor.)—Two radii are drawn at random in a given circle: show (1) how to inscribe symmetrically in the sector thus formed a rectangle of given species; and (2) find the average area of the rectangle, for all forms of the sector. Find also (3) the maximum inscribed rectangle; and (4) the average area of these maximum rectangles.

Solution by Stephen Watson.



1. Construct a rectangle *abcd* of the given species, having its sides *ab*, *cd* perpendicular to the line OL bisecting the angle AOB of the given sector AOB. Through *c*, *d* draw the radii OQ, OP; also draw PS, QR parallel to OL, and join SR. Then it is obvious that *abcd* and PQRS are similar rectangles.

2. Let PQ, RS meet OL in *n*, *m*; put $OA = a$, $\angle AOB = 2\phi$, $mn = x$, $PQ = 2rx$; then we have

$$a^2 = (x + rx\cot\phi)^2 + r^2x^2 = \left\{(1 + r\cot\phi)^2 + r^2\right\}x^2,$$

therefore area of rectangle PQRS $= 2rx^2 = \dfrac{2a^2r}{(1+r\cot\phi)^2+r^2}$ .... (1).

All forms of the sector are given by taking ϕ from 0 to $\frac{1}{2}\pi$, and the measure of total number of rectangles formed is $\frac{1}{2}\pi$; hence the required average is

$$\frac{1}{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(1)\,d\phi = \frac{4a^2r}{\pi}\int_0^{\infty}\frac{dz}{\left\{(1+rz)^2+r^2\right\}(1+z^2)}, \text{ where } z = \cot\phi$$

$$= \frac{4a^2r}{(1+4r^2)\pi}\int_0^{\infty}\left\{\frac{2r^3z+3r^2}{(1+rz)^2+r^2} - \frac{2rz-1}{1+z^2}\right\}dz$$

$$= \frac{4a^2r}{(1+4r^2)\pi}\left(\pi - \cot^{-1}r - r\log\frac{1+r^2}{r^2}\right).$$

[In terms of ϕ, the value of the average is

$$\frac{4a^2r}{\pi}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{(1+r\cot\phi)^2+r^2},$$

which, by integration, is readily found to become

$$\frac{4a^2r}{(1+4r^2)\pi}\left\{\phi + \cot^{-1}\left(\frac{1+r\cot\phi}{r}\right) - r\log\left(\sin^2\phi + r\sin 2\phi + r^2\right)\right\},$$

and this, when ϕ is taken between the limits 0 and $\frac{1}{2}\pi$, gives the same result as that which Mr. WATSON has obtained by his integration in z.

When $r=\frac{1}{2}$, the rectangle becomes a square, and the average is

$$\left(1-\frac{1}{\pi}\tan^{-1}2-\frac{1}{2\pi}\log 5\right)a^2=(\cdot 391)\,a^2.$$

When $r=1$, the length of the rectangle is twice the breadth, and the average is

$$\left(\frac{3}{5}-\frac{4}{5\pi}\log 2\right)a^2=(\cdot 423)\,a^2.$$

3. For any given form of the sector, the inscribed rectangle is a maximum when $r=\sin\phi$, and the maximum area is $a^2\tan\frac{1}{2}\phi$.

4. The average area of these maximum rectangles is

$$\frac{1}{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}a^2\tan\tfrac{1}{2}\phi\,d\phi=\frac{4a^2}{\pi}\log\sec\tfrac{1}{2}\phi=\frac{2a^2}{\pi}\log 2=(\cdot 441)\,a^2.]$$

1100. (Proposed by the EDITOR.)—A straight line is divided at random into three segments; find the probability that it may be possible to form, with these segments, an acute-angled triangle.

Solution by HUGH MCCOLL.

Two points are taken at random in a given straight line, and we are required to find the probability that the square on each segment will be *less* than the sum of the squares of the other two.

Let Q denote the required probability. Let P denote the probability that the square on the *middle* segment will be *greater* than the sum of the squares on the extreme segments. Then it is evident that 3P is the probability that the square on *some one* of the three segments will be *greater* than the sum of the squares on the other two; for the three cases are *à priori* equally probable, and they are mutually exclusive, that is to say, no two of them can co-exist. Hence $Q=1-3P$.

Let OA be the given straight line; let X and Y be the random points in it; and let $OA=a$, $OX=x$, $OY=y$. If the point X falls between A and Y, the three segments will be y, $x-y$, $a-x$; and if it falls between O and Y, the three segments will be x, $y-x$, $a-y$. The probability which we have denoted by P, is thus easily seen to be the probability that *one* of two equally probable and mutually exclusive events will happen; namely, that $(x-y)^2-(a-x)^2-y^2$ will be positive, or else that $(y-x)^2-(a-y)^2-x^2$ will be positive. If, therefore, q denote the probability of the first event, we must have $P=2q$, and therefore $Q=1-6q$.

Now $(x-y)^2-(a-x)^2-y^2$, when simplified, becomes $2ax-2xy-a^2$, and the probability that this expression will be positive when y *alone* is taken at random between 0 and a, is easily seen to be $1-\dfrac{a}{2x}$; and the average value of this chance as x passes from 0 to a is $\dfrac{1}{a}\displaystyle\int_0^a\left(1-\frac{a}{2x}\right)px$.

Hence, $q = \frac{1}{a}\int_0^a \left(1 - \frac{a}{2x}\right) px = \frac{1}{a}\int_{\frac{a}{2}}^a \left(1 - \frac{a}{2x}\right) dx = \frac{1}{2} - \frac{1}{2}\log_e 2;$

and, therefore, $\quad Q = 1 - 6\left(\frac{1}{2} - \frac{1}{2}\log_e 2\right) = 3\log_e 2 - 2.$

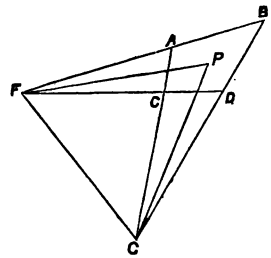
[For the meaning of the symbol px, see Mr. McColl's article on *Probability Notation* on p. 20 of this volume of the *Reprint*. Other solutions of this question, by the Editor and Mr. De Morgan, have been given in the *Reprint*, Vol. II., p. 74, and Vol. V., p. 33.]

1950. (Proposed by Dr. Sylvester.)—If ABCD be a quadrilateral inscribed in a circle, and the sides produced meet at F and G, prove that the bisectors of the angles at F and G meet at right angles.

Solution by C. W. Merrifield, F.R S.

If ABCD be *any* quadrilateral, and FP, GP the bisectors in question, we have

$$\angle \text{PGF} = \tfrac{1}{2}(\text{BGF} + \text{CGF});$$
$$\angle \text{PFG} = \tfrac{1}{2}(\text{BFG} + \text{CFG}).$$



Therefore, between the remaining angles at C, B, and P, we must have

$$\angle \text{FPG} = \tfrac{1}{2}(\text{FCG} + \text{FBG}).$$

But if the quadrilateral be inscribed in a circle, FCG + FBG = two right angles; therefore FPG is a right angle.

[The reciprocal proposition, viz., "In a circumscribed quadrilateral, the bisector of the diagonals passes through the centre of the circle," is due to Newton. An elegant proof of this theorem is given on p. 9 of Paul Serret's *Méthodes en Géométrie;* another, by the aid of *Tangential Coordinates*, will be given by Dr. Booth, the inventor of the method, on another page of this volume of the *Reprint*. Other Solutions of Quest. 1950 are given in the *Reprint*, Vol. V., pp. 105, 106.]

3209. (Proposed by H. McColl.)—Supposing the earth to be a perfect sphere, let l_1 and l_2 denote the latitudes, and λ_1 and λ_2 the longitudes of any two points on the earth's surface, and let θ denote the angle which these points subtend at the earth's centre; show that

$$\cos\theta = \cos(l_1 - l_2) - 2\cos l_1 \cos l_2 \sin^2 \tfrac{1}{2}(\lambda_1 - \lambda_2).$$

Solution by Asher B. Evans, M.A.

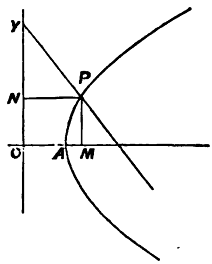
Let P represent one of the poles of the earth, and A and B the two points on its surface; then from the spherical triangle ABP we have

$$\cos AB = \cos AP \cos BP + \sin AP \sin BP \cos P$$
$$= \cos AP \cos BP + \sin AP \sin BP - \sin AP \sin BP (1 - \cos P)$$
$$= \cos (AP - BP) - 2 \sin AP \sin BP \sin^2 \tfrac{1}{2}P \quad \ldots\ldots\ldots\ldots(1).$$

Since $AB = \theta$, and $\lambda_1 - \lambda_2 = P$, equation (1) gives the required result.

3309. (Proposed by C. W. Merrifield, F.R.S.)—Prove that the radius of curvature at any point of a parabola is double the portion of the normal intercepted between the curve and the directrix.

Solution by J. Merrifield, Ph.D., F.R.A.S.; *the* Rev. J. L. Kitchin, M.A.; *and others.*



Let P be any point on the parabola whose coordinates are (x, y); then the equation to the parabola is $y^2 = 4ax$; and the radius (ρ) of curvature at P is

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}} = 2 \frac{(x+a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

Also $$PY = (PN^2 + NY^2)^{\frac{1}{2}}$$
$$= (x+a)\left(1 + \frac{x}{a}\right)^{\frac{1}{2}} = \frac{(x+a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

Hence we have $\rho = 2PY$, which proves the theorem.

3232. (Proposed by G. O. Hanlon.)—From two points A and B on an ellipse four lines are drawn to the foci O, O′; then OA . O′B and O′A . OB are to one another as the squares of the perpendiculars from a focus on the tangents at A and B.

Solution by Stephen Watson.

Let OT, O′T′ and OQ, O′Q′ be the perpendiculars upon the tangents at A and B respectively; then, by similar triangles,

$$OA : O'A = OT : O'T' = OT^2 : OT \cdot O'T' = OT^2 : b^2.$$

Similarly, $\quad OB : O'B = OQ^2 : b^2;$

therefore $\quad OA . O'B : O'A . OB = OT^2 : OQ^2,$

and, by similarity, $\quad = O'T'^2 : O'Q'^2.$

3294. (Proposed by J. F. MOULTON, M.A.)—An indefinitely thin shell of mass m, bounded by two similar and similarly situated and concentric ellipsoids, attracts according to the law of nature. Show that the potential at any external point is given by

$$\frac{dV}{d\lambda} = -\frac{m}{2\left\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\right\}^{\frac{1}{2}}},$$

where λ is the parameter of the ellipsoid confocal with the shell through the point.

Solution by the Rev. R. TOWNSEND, M.A., F.R.S.

Since by well-known formulæ (see Duhamel's *Mécanique*, Vol. I., Art. 142)

$$\frac{dV}{dx} = -\frac{mp_1^2}{a_1b_1c_1}\cdot\frac{x}{a_1^2}, \quad \frac{dV}{dy} = -\frac{mp_1^2}{a_1b_1c_1}\cdot\frac{y}{b_1^2}, \quad \frac{dV}{dz} = -\frac{mp_1^2}{a_1b_1c_1}\cdot\frac{z}{c_1^2},$$

where $\quad a_1^2 = a^2+\lambda, \quad b_1^2 = b^2+\lambda, \quad c_1^2 = c^2+\lambda, \quad \dfrac{1}{p_1^2} = \dfrac{x^2}{a_1^4}+\dfrac{y^2}{b_1^4}+\dfrac{z^2}{c_1^4},$

therefore $\dfrac{dV}{dx}dx + \dfrac{dV}{dy}dy + \dfrac{dV}{dz}dz = dV = -\dfrac{mp_1^2}{a_1b_1c_1}\left(\dfrac{x\,dx}{a_1^2}+\dfrac{y\,dy}{b_1^2}+\dfrac{z\,dz}{c_1^2}\right);$

but since $\dfrac{x^2}{a_1^2}+\dfrac{y^2}{b_1^2}+\dfrac{z^2}{c_1^2} = 1,$ therefore

$$\left(\frac{x\,dx}{a_1^2}+\frac{y\,dy}{b_1^2}+\frac{z\,dz}{c_1^2}\right) = \frac{1}{2}\left(\frac{x^2}{a_1^4}+\frac{y^2}{b_1^4}+\frac{z^2}{c_1^4}\right)d\lambda,$$

therefore $\dfrac{dV}{d\lambda} = -\dfrac{1}{2}\dfrac{m}{a_1b_1c_1},$ and therefore, &c.

Or more shortly thus:—Since

$$p_1^2 = (a^2+\lambda)\cos^2\alpha_1 + (b^2+\lambda)\cos^2\beta_1 + (c^2+\lambda)\cos^2\gamma_1,$$

where $\alpha_1, \beta_1, \gamma_1$ are the direction angles of p_1, therefore $2p_1dp_1 = d\lambda$, and therefore $\dfrac{dV}{d\lambda} = \dfrac{1}{2}\dfrac{1}{p_1}\dfrac{dV}{dp_1}$; but $-\dfrac{dV}{dp_1}$ = the entire resultant attraction of m at (x, y, z), $= \dfrac{p_1m}{a_1b_1c_1}$ (see Duhamel, Vol. I., Art. 141), therefore, as before, $\dfrac{dV}{d\lambda} = -\dfrac{1}{2}\dfrac{m}{a_1b_1c_1}$, and therefore, &c.

NOTE.—Since $\dfrac{a_1b_1c_1}{p_1} = \mu_1\nu_1$, where μ_1 and ν_1 are the semiaxes of the central section of the confocal through (x, y, z) to which p_1 is perpendicular,

it follows immediately from the preceding value for the attraction of the shell, viz., $\frac{mp_1}{a_1 b_1 c_1}$ or $\frac{m}{\mu_1 \nu_1}$, that the attraction of an infinitely thin ellipsoidal shell bounded by similar and parallel surfaces on an external point P is the same as if its entire mass m were concentrated at the point Q on the normal to the confocal ellipsoid through P, for while $\overline{PQ}^2$ = the rectangle under the semiaxes of the central section of that surface to which PQ is perpendicular.

3108. (Proposed by J. J. Walker, M.A.)—Through B, C, angles of a spherical triangle ABC, draw arcs perpendicular to AB, AC respectively, and meeting in D. Let AE, AF be arcs equally inclined to AB, AC, and meeting, the former the base BC in E, the latter a perpendicular arc through D in F; prove that

$$\tan AE \tan AF = \tan b \tan c \quad \text{.........................} \quad (1),$$

$$\tan^2 AD = \frac{\tan^2 b + \tan^2 c - 2 \tan b \tan c \cos A}{\sin^2 A} \quad \text{............} \quad (2).$$

Solution by the Proposer.

1. Draw AG perpendicular to BC. Let $AD = d$, $AE = e$, $AF = f$, $AG = p$. Then $\cos CAD = \tan b \cot d$, $\cos BAD = \tan c \cot d$,

therefore $\frac{\cos CAD}{\cos BAD} = \frac{\tan b}{\tan c} = \frac{\cos BAG}{\cos CAG}$, therefore $\angle CAD = BAG$,

therefore $\tan b \cot d = \tan p \cot b$, therefore $\tan p \tan d = \tan b \tan c$.

Again, since $\angle BAE = CAF$, it follows that $\angle EAG = DAF$; but the cosines of these angles are equal to $\tan p \cot e$ and $\tan f \cot d$ respectively,

therefore $\tan e \tan f = \tan p \tan d = \tan b \tan c$.

2. $$\cot^2 p = \frac{1 - \sin^2 p}{\sin^2 p} = \frac{1 - \sin^2 c \sin^2 B}{\sin^2 c \sin^2 B} = \frac{1 + \cot^2 B - \sin^2 c}{\sin^2 c}$$

$$= \frac{\cos^2 c + \cot^2 B}{\sin^2 c} = \frac{\cos^2 c \sin^2 A + (\sin c \cot b - \cos c \cos A)^2}{\sin^2 c \sin^2 A}$$

$$= \frac{\cot^2 b + \cot^2 c - 2 \cot b \cot c \cos A}{\sin^2 A}.$$

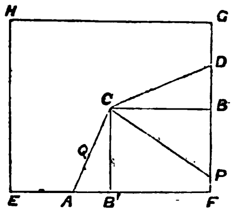
But, from above, $\tan^2 d = \tan^2 b \tan^2 c \cot^2 p$,

therefore $$\tan^2 d = \frac{\tan^2 b + \tan^2 c - 2 \tan b \tan c \cos A}{\sin^2 A}.$$

[Mr. Walker remarks that he is not aware of the above expression for $\cot p$ in terms of b, c, A having been given before.]

3264. (Proposed by H. McColl.)—From the centre C of a square three straight lines are drawn in a random manner as follows:—The first is drawn in a random direction and meets the perimeter of the square in D; the second is drawn to a point P taken at random in the perimeter of the square; and the third is drawn through a point taken at random in the area of the square, and meets the perimeter in A. Determine the respective probabilities of CD, CP, CA being the greatest of the three.

Solution by Stephen Watson; *the* Proposer; *and others.*



Let EFGH be the square, and draw CB, CB′ parallel to EF, FG respectively. The chance will be the same whatever length CB may be; put therefore CB = 1, DCB = ϕ, BP = x, B′A = y; then the number of positions the line CD and the points P and Q can take is $= 2\pi \times 8 \times 4 = 64\pi$.

When CD is greatest, x and y must be each less than $\tan\phi$; hence P and Q can take $8\tan\phi \times 4\tan\phi = 32\tan^2\phi$ positions for one position of CD. The limits of ϕ are 0 and $\frac{1}{4}\pi$, and the result multiplied by 8; hence

$$p_1 = \frac{4}{\pi}\int_0^{\frac{1}{4}\pi} \tan^2\phi\, d\phi = \frac{4}{\pi} - 1.$$

In like manner, when CP is the greatest, the chance is

$$p_2 = \frac{4}{\pi}\int_0^1 x\tan^{-1}x\, dx = 1 - \frac{2}{\pi};$$

and when CA is greatest, the chance is

$$p_3 = \frac{4}{\pi}\int_0^1 y\tan^{-1}y\, dy = 1 - \frac{2}{\pi};$$

therefore $p_1 + p_2 + p_3 = 1$, as it ought to be.

3285. (Proposed by the Rev. J. Wolstenholme, M.A.) — If $s \equiv u + x + y + z$, prove that the determinant

$$\begin{vmatrix} (s-u)^2, & x^2, & y^2, & z^2 \\ u^2, & (s-x)^2, & y^2, & z^2 \\ u^2, & x^2, & (s-y)^2, & z^2 \\ u^2, & x^2, & y^2, & (s-z)^2 \end{vmatrix}$$

$$= 2(u+x+y+z)^5\, uxyz \left\{ \frac{1}{u} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{4}{u+x+y+z} \right\}.$$

I. *Solution by* SAMUEL ROBERTS, M.A.

In the determinant, we may substitute for three lines, by subtracting the 2nd, 3rd, and 4th from the 1st successively,

$$\begin{matrix} s^2-2su, & -s^2+2sx, & 0\ , & 0 \\ s^2-2su, & 0\ , & -s^2+2sy, & 0 \\ s^2-2su, & 0\ , & 0\ , & -s^2+2sz. \end{matrix}$$

Considering the determinant as derived from a system of linear equations in U, X, Y, Z, we have for the common set of values

$$\mathrm{U}:\mathrm{X}:\mathrm{Y}:\mathrm{Z} = \frac{1}{s^2-2su} : \frac{1}{s^2-2sx} : \frac{1}{s^2-2sy} : \frac{1}{s^2-2sz};$$

and the determinant's value is therefore

$$\frac{(s-u)^2}{s^2-2su} + \frac{x^2}{s^2-2sx} + \frac{y^2}{s^2-2sy} + \frac{z^2}{s^2-2sz},$$

or
$$s^3\left(s + \frac{u^2}{s-2u} + \frac{x^2}{s-2x} + \frac{y^2}{s-2y} + \frac{z^2}{s-2z}\right),$$

when cleared of fractions.

I may remark, that the result is of the same form for a corresponding determinant of any number of lines, and the reduction may be made in the same way, as follows:—

We have
$$s^3\left\{\begin{matrix} s^5-2\Sigma u\,.\,s^4+4\Sigma ux\,.\,s^3-8\Sigma uxy\,.\,s^2+16uxyz\,.\,s \\ +u^2\left(s^3-2\Sigma x\,.\,s^2+4\Sigma xy\,.\,s-8xyz\right)+\&\text{c.}\end{matrix}\right\}.$$

Also
$$s^3(s-\Sigma u)^2 = s^5-2\Sigma u\,.\,s^4+(\Sigma u^2+2\Sigma ux)\,s^3 = 0,$$
$$2\left(\Sigma ux\,.\,\Sigma u - u^2\Sigma x - \ldots\right) = 6\Sigma uxy,$$
$$-4\left(\Sigma uxy\,.\,\Sigma u - u^2\Sigma xy - \ldots\right) = -16uxyz, \quad -8\left(u^2\,.\,xyz+\ldots\right) = -8uxyz\,.\,s.$$

Therefore the value of the determinant is

$$s^3\left(2\Sigma uxy\,.\,s^2-8uxyz\,.\,s\right), \quad \text{or} \quad 2s^5uxyz\left(\frac{1}{u}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{4}{s}\right).$$

II. *Solution by* J. J. WALKER, M.A.

Generally a determinant of the form

$$\begin{vmatrix} a+a', & b, & c, & d \\ a, & b+b', & c, & d \\ a, & b, & c+c', & d \\ a, & b, & c, & d+d' \end{vmatrix} = a'b'c'd' + \Sigma a b'c'd'.$$

If therefore $a, b, c, d, a', b', c', d'$ be replaced respectively by u^2, x^2, y^2, z^2, $s(s-2u)$, $s(s-2x)$, $s(s-2y)$, $s(s-2z)$, the value of the determinant in the question will be

$$s^4(s-2u)(s-2x)(s-2y)(s-2z) + s^3\Sigma u^2(s-2x)(s-2y)(s-2z),$$

or
$$s^4\left\{-s^4+4s^2\Sigma(ux)-8s\Sigma(xyz)+16uxyz\right\}$$
$$+\,s^3\Sigma u^2\left\{s^3-2s^2(x+y+z)+4s(yz+zx+xy)-8xyz\right\}.$$

Now $s^6 = s^4\Sigma u^2 + 2s^4\Sigma(ux)$, and $s^5\Sigma(u^2x) = s^6\Sigma(ux) - 3s^5\Sigma(xyz)$, also $s^4\Sigma(u^2yz) = s^5\Sigma(xyz) - 4s^4uxyz$. By means of these relations, the above value is readily reduced to $2s^5\Sigma(xyz) - 8s^4uxyz$; which is, in effect, the result to be proved.

III. *Solution by the* Rev. G. H. Hopkins, M.A.

The determinant equals

$$\begin{vmatrix} s^2-2su, & x^2, & y^2, & z^2 \\ 0, & (s-x)^2, & y^2, & z^2 \\ 0, & x^2, & (s-y)^2, & z^2 \\ 0, & x^2, & y^2, & (s-z)^2 \end{vmatrix} + u^2 \begin{vmatrix} 1, & x^2, & y^2, & z^2 \\ 1, & (s-x)^2, & y^2, & z^2 \\ 1, & x^2, & (s-y)^2, & z^2 \\ 1, & x^2, & y^2, & (s-z)^2 \end{vmatrix}$$

$$= (s^2-2su) \begin{vmatrix} (s-x)^2, & y^2, & z^2 \\ x^2, & (s-y)^2, & z^2 \\ x^2, & y^2, & (s-z)^2 \end{vmatrix} + u^2 \begin{vmatrix} 1, & x^2, & y^2, & z^2 \\ 1, & (s-x)^2, & y^2, & z^2 \\ 1, & x^2, & (s-y)^2 & z^2 \\ 1, & x^2, & y^2, & (s-z)^2 \end{vmatrix}$$

$$= s(s-2u) \begin{vmatrix} (s-x)^2, & y^2, & z^2 \\ x^2, & (s-y)^2, & z^2 \\ x^2, & y^2, & (s-z)^2 \end{vmatrix} + s^3u^2(s-2x)(s-2y)(s-2z)$$

$$= s^2(s-2u)(s-2x) \begin{vmatrix} (s-y)^2, & z^2 \\ y^2, & (s-z)^2 \end{vmatrix} + s^3u^2(s-2x)(s-2y)(s-2z) + s^3x^2(s-2u)(s-2y)(s-2z)$$

$$= s^4(s-2u)(s-2x)(s-2y)(s-2z) + s^3u^2(s-2x)(s-2y)(s-2z) + s^3x^2(s-2u)(s-2y)(s-2z) + s^3y^2(s-2u)(s-2x)(s-2z) + s^3z^2(s-2u)(s-2x)(s-2y)$$

$$\begin{aligned} = s^3\Big\{ & s^5 - 2s^4(u+x+y+z) + s^3(4ux+4uy+4uz+4xy+4xz+4yz + u^2+x^2+y^2+z^2) \\ & - 2s^2(4xyz+4uyz+4uxz+4uxy+u^2x+u^2y+u^2z+x^2u+x^2y+x^2z + y^2u+y^2x+y^2z+z^2u+z^2x+z^2y) \\ & + 4s(4uxyz+u^2xy+u^2xz+u^2yz+x^2uy+x^2uz+x^2yz + y^2ux+y^2uz+y^2xz+z^2ux+z^2uy+z^2xy) \\ & - 8uxyz(u+x+y+z)\Big\} \end{aligned}$$

$$\begin{aligned} = s^3\Big\{ & s^5 - 2s^5 + s^3[s^2+2(ux+uy+uz+xy+xz+yz)] \\ & - 2s^3(ux+uy+uz+xy+xz+yz) - 2s^2(uyz+uxy+uxz+xyz) \\ & + 4s^2(uyz+uxy+uxz+xyz) - 8s\,uxyz\Big\} \end{aligned}$$

$$= s^3\Big\{2s^2(xyz+uyz+uxz+uxy) - 8s\,uxyz\Big\}$$

$$= 2s^5uxyz\left\{\frac{1}{u} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{4}{s}\right\}.$$

3165. (Proposed by Rev. J. Wolstenholme, M.A.)—If S, S′ be two concentric conics, such that triangles can be inscribed in S, whose sides touch

S′, and ABC be any such triangle, A′, B′, C′, the points of contact with S′, then will Δ ABC : Δ A′B′C′ be a constant ratio; and equal to the ratio of the areas of S, S′ if they are ellipses, and to the ratio of the areas of the triangles cut off by any tangent from the asymptotes if they are hyperbolas.

Solution by the PROPOSER.

If two conics are concentric, they have always one, and (unless similar and similarly situate) only one, system of conjugate diameters common in direction. Let the equations of S, S′ referred to these be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1,$$

and let A, B, C be the angular points of a triangle inscribed in S, and whose sides touch S′ in A′, B′, C′ respectively. Take the coordinates of A to be $a\cos\alpha$, $b\sin\alpha$, and similarly for B, C, A′, B′, C′.

Then, because BC touches S′, we get

$$\frac{a'^2}{a^2}\cos^2\tfrac{1}{2}(\beta+\gamma)+\frac{b'^2}{b^2}\sin^2\tfrac{1}{2}(\beta+\gamma)=\cos^2\tfrac{1}{2}(\beta-\gamma),$$

or $\quad$ A $\cos\beta\cos\gamma$ + B $\sin\beta\sin\gamma$ = C, and two like equations ...... (1),

where $\quad \text{A}=-\frac{a'^2}{a^2}+\frac{b'^2}{b^2}+1,\quad \text{B}=\frac{a'^2}{a^2}-\frac{b'^2}{b^2}+1,\quad \text{C}=\frac{a'^2}{a^2}+\frac{b'^2}{b^2}-1.$

But since $\quad \text{A}\cos\alpha\cos\beta+\text{B}\sin\alpha\sin\beta=\text{C},$

and $\quad \text{A}\cos\alpha\cos\gamma+\text{B}\sin\alpha\sin\gamma=\text{C},$

we must have $\quad \dfrac{\cos\frac{1}{2}(\beta+\gamma)}{\text{A}\cos\alpha}=\dfrac{\sin\frac{1}{2}(\beta+\gamma)}{\text{B}\sin\alpha}=\dfrac{\cos\frac{1}{2}(\beta-\gamma)}{\text{C}}$ ....(2),

whence $\quad \dfrac{1}{\text{A}^2}\cos^2\tfrac{1}{2}(\beta+\gamma)+\dfrac{1}{\text{B}^2}\sin^2\tfrac{1}{2}(\beta+\gamma)=\dfrac{1}{\text{C}^2}\cos^2\tfrac{1}{2}(\beta-\gamma),$

or $\quad \left(\dfrac{1}{\text{B}^2}+\dfrac{1}{\text{C}^2}-\dfrac{1}{\text{A}^2}\right)\cos\beta\cos\gamma+\left(\dfrac{1}{\text{C}^2}+\dfrac{1}{\text{A}^2}-\dfrac{1}{\text{B}^2}\right)\sin\beta\sin\gamma=\dfrac{1}{\text{A}^2}+\dfrac{1}{\text{B}^2}-\dfrac{1}{\text{C}^2},$

and from this equation and the former in $\beta\gamma$ we can find $\cos\beta\cos\gamma$ and $\sin\beta\sin\gamma$; but in exactly the same way we could have the same values for $\cos\alpha\cos\beta$, $\sin\alpha\sin\beta$, or for $\cos\alpha\cos\gamma$, $\sin\alpha\sin\gamma$; hence, in order that A, B, C may be distinct points of the conic, the two equations must be coincident, or

$$\frac{1}{\text{A}}\left(\frac{1}{\text{B}^2}+\frac{1}{\text{C}^2}-\frac{1}{\text{A}^2}\right)=\frac{1}{\text{B}}\left(\frac{1}{\text{C}^2}+\frac{1}{\text{A}^2}-\frac{1}{\text{B}^2}\right)=\frac{1}{\text{C}}\left(\frac{1}{\text{A}^2}+\frac{1}{\text{B}^2}-\frac{1}{\text{C}^2}\right),$$

which are both satisfied if $\dfrac{1}{\text{A}}+\dfrac{1}{\text{B}}+\dfrac{1}{\text{C}}=0$, and cannot otherwise be satisfied unless A = B = C, which gives S, S′ coincident. Also this, on substituting for A, B, C their values, leads to the well-known condition $\dfrac{a'}{a}\pm\dfrac{b'}{b}\pm1=0$ for the possibility of such triangles.

(The solution obtained by taking two of the three α, β, γ equal is the

vanishing triangle PQ, if PQ be tangent to S′ at P, a common point, two of the points coinciding at Q and P being the third.)

Now the ratio of the areas of the triangles A′B′C′, ABC is

$$2\cos\tfrac{1}{2}(\beta'-\gamma')\cos\tfrac{1}{2}(\gamma'-\alpha')\cos\tfrac{1}{2}(\alpha'-\beta')\,0:1;$$

and since the tangent at A′ passes through B, C,

$$\frac{a}{a'}\cos\beta\cos\alpha'+\frac{b}{b'}\sin\beta\sin\alpha'=1=\frac{a}{a'}\cos\gamma\cos\alpha'+\frac{b}{b'}\sin\gamma\sin\alpha',$$

therefore $$\frac{\cos\frac{1}{2}(\beta+\gamma)}{\frac{a}{a'}\cos\alpha'}=\frac{\sin\frac{1}{2}(\beta+\gamma)}{\frac{b}{b'}\sin\alpha'}=\cos\tfrac{1}{2}(\beta-\gamma);$$

and comparing these with (2), we have

$$\frac{a}{a'}\cos\alpha'=\frac{A}{C}\cos\alpha,\quad \frac{b}{b'}\sin\alpha'=\frac{B}{C}\sin\alpha.$$

But supposing the relation between the axes to be

$$\frac{a'}{a}+\frac{b'}{b}=1,\quad A=\frac{2b'}{b},\quad B=\frac{2a'}{a},\quad C=-\frac{2a'b'}{ab};$$

therefore $$\frac{A}{C}=-\frac{a}{a'},\quad \frac{B}{C}=-\frac{b}{b'},$$

or $\cos\alpha'=-\cos\alpha$, $\sin\alpha'=-\sin\alpha$, that is, $\alpha'-\alpha=\beta'-\beta=\gamma'-\gamma=\pi$;

therefore $2\cos\frac{1}{2}(\beta'-\gamma')\cos\frac{1}{2}(\gamma'-\alpha')\cos\frac{1}{2}(\alpha'-\beta')$

$$=2\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha)\cos\tfrac{1}{2}(\alpha-\beta)=2\,\frac{(1+qr)(1+rp)(1+pq)}{(1+p^2)(1+q^2)(1+r^2)},$$

if $p\equiv\tan\frac{1}{2}\alpha$, &c.

Now consider the system of equations

$$\cot\tfrac{1}{2}\beta\cot\tfrac{1}{2}\gamma+\cot\tfrac{1}{2}\gamma\cot\tfrac{1}{2}\alpha+\cot\tfrac{1}{2}\alpha\cot\tfrac{1}{2}\beta$$
$$=\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma+\tan\tfrac{1}{2}\gamma\tan\tfrac{1}{2}\alpha+\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\beta=m.$$

This gives the relation between β and γ,

$$(\cot\tfrac{1}{2}\beta+\cot\tfrac{1}{2}\gamma)(\tan\tfrac{1}{2}\beta+\tan\tfrac{1}{2}\gamma)=(m-\cot\tfrac{1}{2}\beta\cot\tfrac{1}{2}\gamma)(m-\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma),$$

or

$$1-m^2+\cot\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma+\cot\tfrac{1}{2}\gamma\tan\tfrac{1}{2}\beta+m\cot\tfrac{1}{2}\beta\cot\tfrac{1}{2}\gamma+m\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma=0,$$

or $$(1-m^2)\sin\beta\sin\gamma+(1+\cos\beta)(1-\cos\gamma)+(1+\cos\gamma)(1-\cos\beta)$$
$$+m\left\{(1+\cos\beta)(1+\cos\gamma)+(1-\cos\beta)(1-\cos\gamma)\right\}=0,$$

or $$2(m-1)\cos\beta\cos\gamma+(1-m^2)\sin\beta\sin\gamma+2(1+m)=0;$$

and comparing this system with (1), we find that they coincide if

$$\frac{A}{2(m-1)}=\frac{B}{1-m^2}=\frac{-C}{2(m+1)}\ \text{(satisfying the condition } \frac{1}{A}+\frac{1}{B}+\frac{1}{C}=0),$$

or if $$m=\frac{C-A}{C+A}=\frac{\frac{-a'}{a}-1}{\frac{-a'}{a}+1},$$ under the same supposition as before.

But if $$qr+rp+pq = \frac{1}{qr}+\frac{1}{rp}+\frac{1}{pq} = m,$$

$$(1+qr)(1+rp)(1+pq) = 1+m+mp^2q^2r^2+p^2q^2r^2 = (1+m)(1+p^2q^2r^2),$$

and $(1+p^2)(1+q^2)(1+r^2)$
$$= 1+(p+q+r)^2-2m+(qr+rp+pq)^2-2mp^2q^2r^2+p^2q^2r^2$$
$$= (1-2m+m^2)(1+p^2q^2r^2),$$

or the ratio of the areas is

$$2(1+m):(1-m)^2, \quad \text{or} \quad \frac{4C}{C+A}:\frac{4A^2}{(C+A)^2}, \quad \text{or} \quad C(C+A):A^2.$$

Since $BC+CA+AB = 0$, this ratio may be written $AB:(A+B)^2$,

which is $$1-\left(\frac{a'^2}{a^2}-\frac{b'^2}{b^2}\right)^2 : 4.$$

But $$\frac{a'^4}{a^4}+\frac{b'^4}{b^4}+1-\frac{2a'^2b'^2}{a^2b^2}-\frac{2a'^2}{a^2}-\frac{2b'^2}{b^2} = 0,$$

or this ratio is $$2\left(1-\frac{a'^2}{a^2}-\frac{b'^2}{b^2}\right):4.$$

Again, $\left(\frac{a'}{a}\pm\frac{b'}{b}\right)^2 = 1$, therefore $1-\frac{a'^2}{a^2}-\frac{b'^2}{b^2} = \pm\frac{2a'b'}{ab}$;

whence, in all cases, $\triangle A'B'C' : \triangle ABC = a'b' : ab$.

If the conics be both ellipses, this ratio is that of their areas; but if both be hyperbolas, it is the ratio of the constant triangles intercepted by their asymptotes and any tangent. It is obvious that the relation cannot be satisfied if one conic be an ellipse and the other a hyperbola.

2655. (Proposed by R. Tucker, M.A.)—With a given major axis all possible ellipses are described; show that for a given abscissa the centres of curvature lie on a certain cubic. Find the envelope of this for different values of the abscissa.

Solution by the Proposer.

The coordinates of the centre of curvature are

$$\alpha = ae^2\cos^3\phi, \quad \beta = -\frac{a^2e^2\sin^3\phi}{b},$$

where ϕ is the eccentric angle. Hence, eliminating e^2 and b, we have for the curve required $a\beta^2\cos^6\phi - a\alpha^2\sin^6\phi = \alpha\beta^2\cos^3\phi$ ....(1),

or changing to x and y, $$y^2 = \frac{ax^2\sin^6\phi}{a\cos^6\phi - x\cos^3\phi}.$$

The curve passes through origin and has an asymptote $x = a\cos^3\phi$.

To find the envelope, we have, differentiating,

$$2a\beta^2\cos^4\phi + 2a\alpha^2\sin^4\phi = a\beta^2\cos\phi;$$

and we have to eliminate λ (where $\lambda = \cos\phi$) between

$$a\lambda^6(x^2+y^2) - 3a\lambda^4x^2 - xy^2\lambda^3 + 3ax^2\lambda^2 - ax^2 = 0,$$

$$2a\lambda^4(x^2+y^2) - 4a\lambda^2x^2 - xy^2\lambda + 2ax^2 = 0,$$

or between $2a\lambda^3 - \lambda^2x - x = 0$ and $2a\lambda^4x + \lambda^3y^2 - 4a\lambda^2x + 2ax = 0$,

or between $2a\lambda^3 - \lambda^2x - x = 0$ and $\lambda^2(x^2+y^2+4a^2) - 6\lambda ax + x^2 = 0$;

whence we obtain for the required equation

$$y^4(x^2+y^2) + 48a^4y^2 + 80a^2x^2y^2 + 12a^2y^4 + 64a^2(x^2-a^2)^2 = 0.$$

3280. (Proposed by Professor FICKLIN.)—A plank, upon which at the upper end a dog is standing, is placed directly along a smooth inclined plane. It is required to determine how long it will take the dog to run down the plank, so that it may not stir till he is off it.

Solution by W. GRIFFITH; W. SIVERLY; REV. J. L. KITCHIN, M.A.; *and others.*

Let a be the length of the plank, θ the angle of inclination of the plane, m and m' the masses of the dog and plank respectively, and x the distance passed over by the dog in any given time t.

The tendency of the plank, with the dog on it, to slide down the plane is $(m+m')g\sin\theta$.

The impulsive force, which must be resolved parallel to the plane, equals this, as is evident from the conditions of the question.

We shall then have the following equation:

$$\frac{m'd^2x}{dt^2} = (m+m')g\sin\theta.$$

By integrating twice, and putting $x = a$, we have

$$t = \left(\frac{2am'}{(m+m')g\sin\theta}\right)^{\frac{1}{2}}.$$

3282. (Proposed by W. K. CLIFFORD, M.A.)—It is known that the circles circumscribing the triangles formed by 4 lines meet in a point, and that the points so belonging to the 5 tetragrams formed by 5 lines lie in a circle. Prove that the circles so belonging to the 6 pentagrams formed by 6 lines meet in a point, and so on; the series of theorems being interminable.

To every $2n+1$ lines there belongs in this way a circle. If from any point p on this circle perpendiculars be let fall on the straight lines, their feet will all lie on a curve of order n, having a $(n-1)$-ple point at p.

Solution by C. FUORTES.

On démontre la généralité du théorème, en prouvant que, s'il est vrai pour le système de $2n-1$ droites, il est encore vrai pour les systèmes de $2n$ et de $2n+1$ droites.

Je parle, pour plus de clarté, sur de nombres particulaires, mais le raisonnement est tout-à-fait général.

Je suppose donc le théorème vérifié pour le système de cinq droites. C'est-à-dire, je suppose que—(1). Dans un système de cinq droites 1, 2, 3, 4, 5, les cinq points p_1 (2345), p_2 (1345), p_3 (1245), p_4 (1235), p_5 (1234), correspondants aux cinq quadrilatères, qu'on obtient par les droites du système, sont sur un cercle, que j'apelle C (12345). (2). Ce cercle C (12345) est le lieu d'un point tel qu'en abaissant les perpendiculaires du point sur les cinq droites, les pieds de ces perpendiculaires appartiennent à une conique passant par le point. Cela posé, je dis que—

1. Le théorème est vrai pour le système de six droites 1, 2, 3, 4, 5, 6.

Je cherche, en effet, un point m, dans le plan du système, tel qu'en abaissant par m les perpendiculaires sur les six droites, les pieds des perpendiculaires appartiennent à une conique passant par m. Ce point évidemment doit se trouver sur deux quelconques des six cercles correspondants aux pentagones formés par les droites données, par exemple sur les cercles C_1 (23456), C_2 (13456). Or les points communs à ces cercles sont le point p (3456) correspondant au quadrilatère 3456, un autre point P, et les deux points circulaires à l'infini. Le point p (3456) ne répond pas à la question, car il dépend seulement des quatre droites 3, 4, 5, 6; pour cela P à distance finie, et les points circulaires à l'infini, sont les seules positions du point m cherché.

Il suit de là que tous les cercles, correspondants aux pentagones, qu'on obtient par cinq des six droites du système, passent par le point P. Donc est vrai.

2. Le théorème est vrai pour le système de sept droites 1, 2, 3, 4, 5, 6, 7.

Je cherche, dans le plan du système, le lieu du point tel qu'en abaissant les perpendiculaires sur les sept droites, les pieds de ces perpendiculaires, soient sur une cubique qui ait un point double dans le point du lieu.

Le lieu cherché a sur le cercle C (34567), correspondant à une des pentagones des sept droites, quatre points, savoir: les deux points p_1 (234567) et p_2 (134567) correspondants à deux des exagones formés par les droites du système, et les deux points circulaires à l'infini. Car les perpendiculaires abaissées d'un quelconque de ces points sur six droites du système, ont les pieds sur une conique passant par le point, laquelle avec la septième perpendiculaire forme une courbe du troisième ordre passant deux fois par le point. Il est en outre évident qu'aucun point du cercle C, hors les quatre nommés, appartient au lieu, qui par conséquence est du second ordre et passe par les points circulaires à l'infini. Il est donc un cercle sur lequel se trouvent les points correspondants aux exagones du système de droites. Donc, etc.

2516. (Proposed by the EDITOR.)—In the sides BC, CA, AB of a triangle ABC, points D, E, F are taken such that

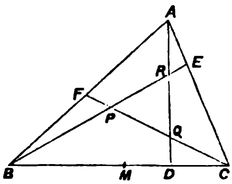
$$BD : CD = CE : AE = AF : BF = l : m.$$

Find the average area (1) of the triangle (Δ_1) whose sides are equal to the

lines AD, BE, CF, and (2) of the triangle (Δ_2) whose vertices are the intersections P, Q, R of the lines AD, BE, CF. Find also the respective probabilities (p_1, p_2) that, if D be taken at random in BC, (3) the triangle (Δ_1), or (4) the triangle (Δ_2), will be less than the average area.

Solution by S. Bills; A. Martin; *and others.*

1. Let M be the middle point of BC, and put MB = MC = a, and MD = x; then BD = $a+x$ and CD = $a-x$. Thus, if Δ be the area of ABC, we have, by the solution of Quest. 2338 (*Reprint*, Vol. VII., pp. 89—91),



$$\Delta_1 = \frac{l^2+lm+m^2}{(l+m)^2}\Delta$$

$$= \frac{(a+x)^2+(a+x)(a-x)+(a-x)^2}{4a^2}\Delta = \frac{x^2+3a^2}{4a^2}\Delta.$$

Hence $\int_{-a}^{+a} \frac{x^2+3a^2}{4a^2}\Delta dx = \frac{5}{3}a\Delta$ expresses the sum of the areas, while D ranges from B to C. Dividing the above by $2a$, the measure of the number of triangles, we have $\frac{5}{6}\Delta$ as the average area required in this case.

2. Here, by Question 2338, we have

$$\Delta_2 = \frac{(l-m)^2}{l^2+lm+m^2}\Delta = \frac{4x^2}{x^2+3a^2}\Delta;$$

and $\int_{-a}^{+a} \frac{4x^2}{x^2+3a^2}\Delta dx = \frac{4}{3}(6-\pi\sqrt{3})\,a\Delta$ expresses the sum of the areas, while D ranges from B to C; and dividing by $2a$, the average area is

$$\tfrac{2}{3}(6-\pi\sqrt{3})\,\Delta.$$

3. Suppose, in the first case, that Δ_1 is not to exceed $n\Delta$, where n is any fraction from $\frac{3}{4}$ to 1. Put $\Delta_1 = n\Delta$; then $\frac{x^2+3a^2}{4a^2} = n$; whence $x = \pm a(4n-3)^{\frac{1}{2}}$, and the probability required is

$$p_1 = 2a(4n-3)^{\frac{1}{2}} \div 2a = (4n-3)^{\frac{1}{2}}.$$

Giving n the value $\frac{5}{6}$, the average in this case, we have $p_1 = \frac{1}{3}\sqrt{3}$.

4. Suppose the triangle in the second case is not to exceed $n\Delta$, where n is any fraction from 0 to 1; then we have $\frac{4x^2}{x^2+3a^2} = n$; whence $x = \pm a\left(\frac{3n}{4-n}\right)^{\frac{1}{2}}$, and the probability required is

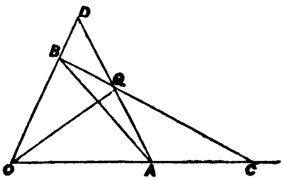
$$p_2 = 2a\left(\frac{3n}{4-n}\right)^{\frac{1}{2}} \div 2a = \left(\frac{3n}{4-n}\right)^{\frac{1}{2}}.$$

Giving n the value $\frac{2}{3}(6-\pi\sqrt{3})$, the average in this case, we have

$$p_2 = \left(\frac{6\sqrt{3}-3\pi}{\pi}\right)^{\frac{1}{2}}.$$

[In parts of the area of the triangle ABC, the average areas of Δ_1, Δ_2 are respectively ·8333, ·3724; and the respective probabilities that the random-formed triangles are less than the average values are ·5774, ·5549; moreover, it is an even chance ($p_1 = p_2 = \frac{1}{2}$) that the random-formed triangle Δ_1 will be less than $\frac{13}{16}\Delta$, and Δ_2 less than $\frac{4}{13}\Delta$.]

A Solution of Newton's Theorem by the Method of Tangential Coordinates. *By the* Rev. Dr. Booth, F.R.S.



A simple solution is not easily found by the ordinary methods for Newton's theorem, that the line which joins the middle points of the diagonals of a quadrilateral passes through the centre of an inscribed conic.

Let OQAB be the quadrilateral, let OA $= a$, OC $= c$, OB $= b$, OD $= d$.

Now the general tangential equation of a conic section is

$$\alpha\xi^2 + \alpha_1 v^2 + 2\beta\xi v + 2\gamma\xi + 2\gamma_1 v = 1 \quad\text{.....................}\quad (1),$$

where γ and γ_1 *always* denote the projective coordinates of the centre of the curve.

Let O be taken as origin, and OA, OB as axes of coordinates. Then, as the curve touches the axis of X, $\alpha_1 = 0$, and as it touches the axis of Y, $\alpha = 0$; hence the equation is now reduced to

$$2\beta\xi v + 2\gamma\xi + 2\gamma' v = 1 \quad\text{...........................}\quad (2).$$

Now as the line AD is a tangent to the curve, $a = \dfrac{1}{\xi}$, and $d = \dfrac{1}{v}$; substituting these values in (2), we get

$$2\beta + 2d\gamma + 2a\gamma_1 = ad \quad\text{..............................}\quad (3);$$

and as CB is a tangent to the curve, $c = \dfrac{1}{\xi}$, $b = \dfrac{1}{v}$; hence, substituting in

(2), we have $$2\beta + 2b\gamma + 2c\gamma_1 = cb \quad\text{..............................}\quad (4).$$

Subtracting the former from the latter, we find

$$2(b-d)\gamma + 2(c-a)\gamma_1 = cb - ad \quad\text{.....................}\quad (5),$$

a condition which proves that the centres of the inscribed conic sections always move along a straight line.

We shall now find the equation of the line which joins the middle points of the diagonals. The coordinates of the middle point of AB are evidently $y' = \frac{1}{2}b$, $x' = \frac{1}{2}a$; also the equations of the lines AD, CB are

$$\frac{x}{a} + \frac{y}{d} = 1, \quad \frac{x}{c} + \frac{y}{b} = 1;$$

hence the projective coordinates of Q, their point of intersection, are

$$y'' = \frac{db\,(a-c)}{ab-cd}, \quad x'' = \frac{ac\,(b-d)}{ab-cd} \quad\text{.....................} (6),$$

and the projective coordinates of the middle point of OQ are $(\frac{1}{2}y'', \frac{1}{2}x'')$.

Now the equation of a straight line passing through the points (y', x'), (y'', x'') is

$$y-y' = \frac{y''-y'}{x''-x'}\,(x-x');$$

or, substituting the values found for y', y'', x', x'', we get

$$2\,(b-d)\,x+2\,(c-a)\,y = cb-ad \quad\text{........................} (7),$$

which is identically the same equation as (5).

3314. (Proposed by the EDITOR.)—A King is placed at random on a chess-board, and then (1) a Bishop, or (2) a Knight, or (3) a Rook, or (4) a Queen; find, in each case, the chance that the King is in check.

Solution by the REV. J. L. KITCHIN, M.A.

1	9	17	25	33	41	49	57
2	10	18	26	34	42	50	58
3	11	19	27	35	43	51	59
4	12	20	28	36	44	52	60
5	13	21	29	37	45	53	61
6	14	22	30	38	46	54	62
7	15	23	31	39	47	55	63
8	16	24	32	40	48	56	64

1. The total number of different positions in which King and Bishop can stand upon the board is 64.63. To find the favourable cases, place the King on 1; then the Bishop may stand on any of the seven spaces marked 10, 19 ... 64. In the same way, the King is in check on seven squares for 2, 3 ... 8, or for that row 56 in all. Now the King on 9, 10, 11, 12 is in check on 7, 9, 9, 9 squares respectively; therefore for the whole row the number of favourable cases is 68. The King on 17, 18, 19, 20 is in check on 7, 9, 11, 11 squares respectively; therefore for the whole row the number of favourable cases is 76. The King on 25, 26, 27, 28 is in check on 7, 9, 11, 13 squares respectively; therefore for the whole row the number of favourable cases is 80.

The other half of the board is but a repetition of these four rows; therefore, adding and multiplying by 2, the favourable cases are in all 560. Hence the chance of check $= \dfrac{560}{64.63} = \dfrac{5}{36}$.

2. In the same way, for the Knight, we find the chance $= \dfrac{336}{64.63} = \dfrac{1}{12}$.

[3. For a Rook, the chance is $\dfrac{14}{63}$, or $\dfrac{2}{9}$; since, on whatever square the King stands, the Rook will check it on 14 squares.

4. For the Queen, the chance is, evidently, $\frac{5}{36}+\frac{2}{9}=\frac{13}{36}$, being the sum of the chances of the Bishop and Rook.

Hence the relative *checking* powers of the pieces, on a clear board, are

Queen's : Rook's : Bishop's : Knight's = 13 : 8 : 5 : 3.

The *entire* relative powers of the pieces, as given in the *Chess Player's Magazine* (Vol. II., for 1866, p. 165), are

Queen's : Rook's : Bishop's : Knight's = $9\frac{1}{2}$: $4\frac{1}{2}$: $3\frac{1}{2}$: 3.]

3343. (Proposed by the Rev. J. Wolstenholme, M.A.)—If curves be described similar and similarly situate to a given curve U, and having contact of the second order with any curve V at any point P, the locus of any point S similarly situate in the moving curve to a fixed point O in U will touch PS at S.

I. *Solution by* J. F. Moulton, M.A.

For we may conceive two curves similar to U as the limits of curves passing through points OPQ, PQR respectively of the curve V; hence they have the common chord PQ becoming in the limit a common tangent. But when two similar and similarly situated curves touch, the point of contact is a centre of similarity; and two consecutive positions of S will therefore lie on a straight line through this point of contact; that is, the straight line PS is a tangent at S to the locus of S.

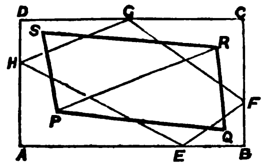
II. *Solution by* F. D. Thomson, M.A.

We may regard the osculating curve as having three consecutive points in common with the fixed curve V. Hence two consecutive osculating curves have two points common, or, in other words, have a common tangent at the point P.

Now, in similar and similarly situated curves, tangents at corresponding points are parallel. Hence the point P is in the same position relatively to each curve. Therefore, if PS be drawn to any point S in the one curve, it will meet the consecutive curve in a point S′ which is similarly situated to the point S in the other. In other words, the locus of S touches PS.

3188. (Proposed by S. Watson.)—On the respective sides AB, BC, CD, DA of a given rectangle ABCD, points E, F, G, H are taken at random, and EF, FG, GH, HE joined. Again, in the respective triangles HAE, EBF, FCG, GDH, points P, Q, R, S are taken at random. Show that the average area of the quadrilateral PQRS is four-ninths of the parallelogram.

Solution by the PROPOSER.



Take AB, AD as axes, and put AB $=a$, AD $=b$, AE $=x$, DG $=x'$, AH $=y$, BF $=y'$. Now if any three of the points P, Q, R, S, as Q, R, S, be fixed, while P takes every position in the triangle HAE, the average area of the quadrilateral will be that when P is at the centre of gravity of the triangle HAE. From this it follows that the average area of the quadrilateral PQRS, when P, Q, R, S take every possible position in their respective triangles, is that when P, Q, R, S are at the centres of gravity of the triangles HAE, EBF, FCG, GDH respectively. Moreover, P, Q, R, S being the centres of gravity, as above, x_1y_1, x_2y_2, x_3y_3, x_4y_4 their respective coordinates, we have

$$x_1 = \tfrac{1}{3}x, \quad x_2 = \tfrac{1}{3}(2a+x), \quad x_3 = \tfrac{1}{3}(2a+x'), \quad x_4 = \tfrac{1}{3}x',$$
$$y_1 = \tfrac{1}{3}y, \quad y_2 = \tfrac{1}{3}y', \quad y_3 = \tfrac{1}{3}(2b+y'), \quad y_4 = \tfrac{1}{3}(2b+y);$$

hence the area of the triangle PQR is

$$\tfrac{1}{2}\left\{y_1(x_3-x_2)+y_2(x_1-x_3)+y_3(x_2-x_1)\right\} = \tfrac{2}{9}ab-\tfrac{1}{18}(y-y')(x-x').$$

In like manner, $\triangle$ RSP $= \tfrac{2}{9}ab-\tfrac{1}{18}(x-x')(y-y')$.

Also the area of the quadrilateral EFGH $= \tfrac{1}{2}ab-\tfrac{1}{2}(x-x')(y-y')$;

therefore the quadrilateral PQRS $= \tfrac{1}{3}ab+\tfrac{2}{9}$ quadrilateral EFGH;

but the average area of the quadrilateral EFGH is $\tfrac{1}{2}ab$,

therefore the average area of the quadrilateral PQRS is $\tfrac{4}{9}ab$.

It may be easily shown that PQRS is a parallelogram.

3176. (Proposed by S. ROBERTS, M.A.)—A line of given length moves with one extremity on a sphere, and the other on a line; to find the locus of a point on the moving line.

Solution by the PROPOSER.

Let x_1, y_1, z_1 be the coordinates of one extremity of the moving line; and let $y=0$, $z=0$ be the equations of the rule line.

The conditions of the question may be written in the form

$$(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2 = r^2,$$
$$(x_1-x_2)^2+y_1^2+z_1^2 = (m+n)^2,$$
$$x = \frac{mx_2+nx_1}{m+n}, \quad y = \frac{ny_1}{m+n}, \quad z = \frac{nz_1}{m+n}.$$

From $\quad mx_2+nx_1 = (m+n)x, \quad x_1-x_2 = \left\{(m+n)^2-y_1^2-z_1^2\right\}^{\frac{1}{2}},$

we get $\quad (m+n)x_1 = (m+n)x+m\left\{(m+n)^2-y_1^2-z_1^2\right\}^{\frac{1}{2}}.$

Therefore the equation of the locus is

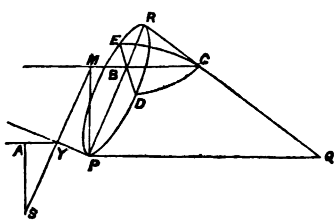
$$\left\{x-a+m\left(1-\frac{y^2+z^2}{n^2}\right)^{\frac{1}{2}}\right\}^2+\left(\frac{m+n}{n}y-\beta\right)^2+\left(\frac{m+n}{n}z-\gamma\right)=r^2,$$

representing a surface of the fourth degree.

It is noticeable that the sections of the surface by planes through the rule line are circular quartics, which can be continuously traced; and sections by planes perpendicular to the rule line are a class of bicircular quartics.

2983. (Proposed by W. H. H. HUDSON, M.A.)—A cone filled with fluid is placed with a generating line horizontal, and the axis in a vertical plane above it; and a small hole is made at the lowest point of the base. Find the time of emptying, and the mean value of the latus rectum of the issuing jet.

Solution by F. D. THOMSON, M.A.



1. Let BDCE be the parabolic section of the cone by the surface of the fluid at any time t; α the semi-vertical angle of the cone; a the radius of the base.

Then the vertical velocity at the surface is $\frac{d}{dt}(\text{RB}\cos\alpha)$, or $\frac{dx}{dt}\cos\alpha$, if we write x for RB.

Also the velocity at the orifice is approximately $\left\{2g(2a-x)\cos\alpha\right\}^{\frac{1}{2}}$, and the area of the section DCE is $\frac{4}{3}\text{BD}\,.\,\text{BC}=\frac{2}{3}x\operatorname{cosec}\alpha\left\{x(2a-x)\right\}^{\frac{1}{2}}$.

Hence, if K be the area of the orifice,

$$\text{K}\left\{2g(2a-x)\cos\alpha\right\}^{\frac{1}{2}}=\tfrac{2}{3}x\operatorname{cosec}\alpha\left\{x(2a-x)\right\}^{\frac{1}{2}}\frac{dx}{dt}\cos\alpha\,;$$

therefore $$\text{K}(2g\cos\alpha)^{\frac{1}{2}}=\tfrac{2}{3}x^{\frac{3}{2}}\cot\alpha\frac{dx}{dt};$$

therefore $$\text{K}(2g\cos\alpha)^{\frac{1}{2}}t=\tfrac{2}{3}\cot\alpha\int_0^x x^{\frac{3}{2}}\,dx=\tfrac{4}{15}\cot\alpha\,.\,x^{\frac{5}{2}};$$

therefore $$t=\frac{4x^{\frac{5}{2}}\cot\alpha}{15\text{K}(2g\cos\alpha)^{\frac{1}{2}}};$$

therefore $$\text{whole time of efflux}=\frac{16a^2}{15\text{K}}\cot\alpha\left(\frac{a}{g\cos\alpha}\right)^{\frac{1}{2}}.$$

2. Assuming the orifice to be made in the base so that the direction of the jet is perpendicular to the base, as PY, the parabolic path will have CBM for its directrix, and the focus S will be constructed by drawing PM perpendicular to CB, MYS perpendicular to PY, and making YS = MY. Also, if A be the vertex of the parabola,

$$SA \,.\, SP = SY^2 = MY^2 = PM^2 \cos^2\alpha\,;$$

therefore $SA = PM\cos^2\alpha$ (since $SP = PM$) $= BP\cos^3\alpha = (2a-x)\cos^3\alpha$.

Now, as above, we have $x^{\frac{5}{2}} = \frac{15}{4}K\tan\alpha\,(2g\cos\alpha)^{\frac{1}{2}}\,t = c^{\frac{5}{2}}\,t$ suppose;

therefore $\qquad x = ct^{\frac{2}{5}}$ and $SA = (2a - ct^{\frac{2}{5}})\cos^3\alpha\,;$

therefore the mean value of $SA\sec^3\alpha$ during the whole time τ of efflux

$$= \frac{1}{\tau}\int_0^\tau (2a - ct^{\frac{2}{5}})\,dt = 2a - \tfrac{5}{7}c\tau^{\frac{2}{5}}.$$

But $\qquad c^{\frac{5}{2}}\tau = \frac{15}{4}K\tan\alpha\,(2g\cos\alpha)^{\frac{1}{2}}\,\frac{16a^2}{15K}\cot\alpha\left(\frac{a}{g\cos\alpha}\right)^{\frac{1}{2}}$

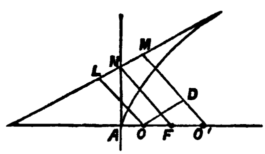
$$= 4a^2(2a)^{\frac{1}{2}} = (2a)^{\frac{5}{2}}\,;$$

therefore $c\tau^{\frac{2}{5}} = 2a$; therefore mean value of $SA\sec^3\alpha = \frac{4}{7}a$;

therefore $\qquad$ mean value of latus rectum $= \frac{16}{7}a\cos^3\alpha$.

3330. (Proposed by Dr. Sylvester.)—If two points be fixed in the axis of a parabola at equal distances from the focus; prove (1) that the difference of the squares on the perpendiculars drawn from them to any tangent is constant. Also (2) invent an analogous theorem applicable to any conic.

I. *Solution by* Benjamin Williamson, M.A.



1. Let O, O′ be the points; OL, O′M the perpendiculars, FN the perpendicular from the focus F. Then

$$O'M^2 - OL^2 = (O'M + OL)(O'M - OL)$$
$$= 2FN \,.\, O'D$$

(OD being parallel to the tangent)

$$= 2OO' \,.\, FA.$$

2. The analogous theorem is this:—If O, O′ be points harmonic conjugate to each other, with respect to the foci of a conic; then, if p, p' represent the perpendiculars from O, O′ on any tangent,

$$p^2 \,.\, FO'^2 - p'^2 \,.\, FO^2 = \text{constant},$$

which admits of an easy analytical proof.

II. *Solution by the* Rev. R. Townsend, M.A., F.R.S.

Letting fall the perpendicular from the focus upon any tangent to the parabola, and drawing through its foot the parallel to the axis to meet the perpendicular from either point on the tangent; it is at once seen, by similar triangles, that the difference of the squares in question is equal to four times the rectangle under the distances of the focus from the vertex and from either point; and therefore &c., as regards the first part of the question.

For any conic, as regards the second part, if the distance EF between the two foci E and F be cut harmonically at any two points P and Q, then, if p, q, e, f be the four perpendiculars from P, Q, E, F upon any tangent to the curve, $mp^2 - nq^2 = (m-n)\,ef$, where $\pm\left(\frac{m}{n}\right)^{\frac{1}{2}}$ are the two ratios of the section of PQ by E and F; which may be readily proved as follows:—Since

$$\sqrt{m}\,.\,p \pm \sqrt{n}\,.\,q = (\sqrt{m} \pm \sqrt{n})\,e, \quad \text{and} \quad \sqrt{m}\,.\,p \mp \sqrt{n}\,.\,q = (\sqrt{m} \mp \sqrt{n})\,f,$$

therefore at once, by multiplication, $mp^2 - nq^2 = (m-n)\,ef$; and therefore &c.

Note.—The converses of the above come evidently, as particular cases, under the following well known general property; viz.—

If A, B, C, D, &c. be any system of fixed points in a plane, and a, b, c, d, &c. any corresponding system of finite multiples having similar or mixed signs, the envelope of a variable line L, for which the sum $\Sigma\,(a\,.\,\overline{\mathrm{AL}}^2) =$ a constant, is an ellipse or hyperbola, whose centre O is the mean centre, whose axes OX and OY are the central principal axes, and whose foci E and F are the two points for which all axes are principal, of the system of points for the system of multiples; and which degenerates into a parabola in the particular case when $\Sigma\,(a) = 0$.

3322. (Proposed by H. M'Coll.)—In the cubic equation $ax^3 + bx + c = 0$, the numerical values of a, b, c are each taken at random between 0 and any positive quantity n, and a random sign ($+$ or $-$) is prefixed to each. Show that the probability of all the roots being real is $\frac{4}{45}\sqrt{3} - \frac{1}{54}$, whatever value we give to n.

I. *Solution by the* Proposer.

All the roots will be real if two independent conditions are satisfied, namely, *if a and b have opposite signs*, and *if $\left(\frac{b}{a}\right)^3$ is numerically greater than $\frac{27}{4}\left(\frac{c}{a}\right)^2$*. The probability of the first condition being evidently $\frac{1}{2}$, if we denote the probability of the second condition by p, the required probability of three real roots will be $\frac{1}{2}p$.

Now, the second condition will be satisfied provided that c is numerically less than $\left(\frac{4b^3}{27a}\right)^{\frac{1}{2}}$, and not otherwise; and the probability that this

will happen when c *alone* is taken at random between the assigned limits is evidently $\frac{1}{n}\left(\frac{4b^3}{27a}\right)^{\frac{1}{2}}$. Hence the probability that the second condition will be satisfied when b and c are taken at random, while a remains constant, is $\frac{1}{n}\int_0^n \frac{1}{n}\left(\frac{4b^3}{27a}\right)^{\frac{1}{2}} pb$; and this becomes $1-\frac{9}{10}\left(\frac{2a}{n}\right)^{\frac{1}{2}} = \phi(a)$, when $n > \left(\frac{27an^2}{4}\right)^{\frac{1}{3}}$, and therefore $a < \frac{4n}{27}$; but the same expression becomes $\frac{4}{45}\left(\frac{3n}{a}\right)^{\frac{1}{2}} = \phi_1(a)$ say, when $a > \frac{4n}{27}$.

Hence $$p = \frac{1}{n}\int_0^{\frac{4n}{27}} \phi(a)\,pa + \frac{1}{n}\int_{\frac{4n}{27}}^{n} \phi_1(a)\,pa = \frac{8}{45}\sqrt{3} - \frac{1}{27};$$

and therefore $$\tfrac{1}{2}p = \frac{4}{45}\sqrt{3} - \frac{1}{54}.$$

[The whole operation might be indicated briefly thus:—
$$\tfrac{1}{2}\mathrm{P} = \frac{1}{2n}\int_0^n \frac{1}{n}\int_0^n \frac{1}{n}\left(\frac{4b^3}{27a}\right)^{\frac{1}{2}} pa\, pb = \frac{4}{45}\sqrt{3} - \frac{1}{54}.$$
For the meaning of the symbols pa, pb, see Mr. M'Coll's article on *Probability Notation*, on p. 20 of this Volume of the *Reprint*.]

II. *Solution by* Stephen Watson.

Writing the equation $lx^3 + mx + n = 0$, the condition for real roots is
$$n - \frac{2}{3\sqrt{3}}\left(\frac{m^3}{l}\right)^{\frac{1}{2}} < 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hence, if 0 and a be the limits between which l, m, n may lie, n may take all values between $\pm a$,

(α.) When l lies between 0 and $\frac{4m^3}{27a^2}$ and m between 0 and a.

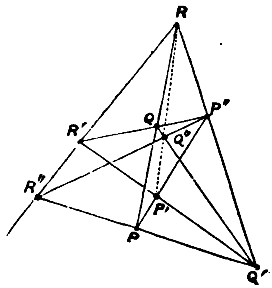
Also n must lie between $\pm\frac{2}{3\sqrt{3}}\left(\frac{m^3}{l}\right)^{\frac{1}{2}}$,

(β.) When l lies between $\frac{4m^3}{27a^2}$ and a and m between 0 and a.

Now (α) and (β) must both be doubled, because in (1) m and l may both be $-$ as well as $+$. The required chance is therefore
$$p = \frac{1}{8a^3}\left\{2\int_0^a dm\left[\int_0^{\frac{4m^3}{27a^2}} 2a\,dl + \int_{\frac{4m^3}{27a^2}}^{a} \frac{4}{3\sqrt{3}}\left(\frac{m^3}{l}\right)^{\frac{1}{2}} dl\right]\right\}$$
$$= \frac{1}{8a^3}\int_0^a \frac{16}{27}\left(3\sqrt{3}\,a^{\frac{1}{2}}m^{\frac{3}{2}} - \frac{m^3}{a}\right)dm = \frac{4}{45}\sqrt{3} - \frac{1}{54}.$$

3355. (Proposed by Dr. SYLVESTER.)—The point of intersection of two straight lines, and also two other points on each of them, being given as five of the flexures of a cubic curve; determine the locus of the remaining four flexures.

Solution by F. D. THOMSON, M.A.



It is evident that if the point of intersection of the two lines is real, one pair at least of the points on each line must be imaginary.

Let R, R′, R″, P, Q be the five given points of inflexion; then, since the line joining any two points of inflexion passes through a third point of inflexion, we may, on assuming P″ to be a sixth point of inflexion on the line R′Q, construct as in the figure and obtain the remaining three points of inflexion P′Q′Q″.

It follows *from the construction* that RQ″P′ is a straight line.

But, by the property above mentioned, R′Q″P, R″QP′ must also be straight lines.

To find the necessary condition, assume $x=0$, $y=0$, $z=0$, $lx+my+nz=0$, $ax+by=0$ as the equations to RR′R″, RQP, R″PQ′, R′QP″, RP″Q′ respectively.

Then the equation to QQ′ is $lax+lby+naz=0$
,, R″P″ is $(lb-ma)x+nbz=0$
,, R′P is $my+nz=0$, which meet in Q″;

therefore
$$\begin{vmatrix} la, & lb, & na \\ lb-ma, & 0, & nb \\ 0, & m, & n \end{vmatrix} = 0,$$
or
$$n\,(l^2b^2-lmab+m^2a^2)=0,$$
the condition required, which would also follow from R″QP′ being a straight line.

But if x, y be the coordinates of P″, $\dfrac{x}{b}=\dfrac{y}{-a}$;

therefore, rejecting the factor n, $l^2x^2+lmxy+m^2y^2=0$,

the equation to two fixed straight lines through the point R. Therefore the four points of inflexion lie in pairs on two straight lines.

3310. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—A rod AB is marked at random in P, Q; and a point R is then taken at random in PQ. Show that $3\log_e 2-2$ is the chance that $PR^2+RQ^2>AP^2+QB^2$.

Solution by STEPHEN WATSON.

Put $AB=a$, $AP=ax$, $AQ=ay$, and $AR=az$; then

$(z-x)^2+(y-z)^2>x^2+(1-y)^2$, therefore $x<\dfrac{z^2+(y-z)^2-(1-y)^2}{2z}$;

and since x must lie between 0 and z, the limits of y are $\dfrac{1-2z}{2(1-z)}$, $\dfrac{1}{2(1-z)}$, which put $= m, n$; also the limits of z are 0 and $\frac{1}{2}$. Hence the chance is

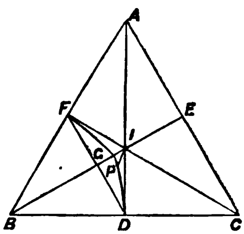
$$p = \int_0^{\frac{1}{2}} dz \int_m^n dy \left\{\frac{z^2+(y-z)^2-(1-y)^2}{2z}\right\} \div \int_0^1 dz \int_z^1 dy \int_0^z dx$$

$$= 3\int_0^{\frac{1}{2}} \frac{z^2dz}{1-z} = 3\log_e 2-2.$$

3342. (Proposed by H. McColl.)—A point is taken at random inside an equilateral triangle, and from it perpendiculars are drawn to the sides. Show that $3\log_e 2-2$ is the chance that straight lines equal respectively to these three perpendiculars can be the sides of an acute-angled triangle.

Solution by Stephen Watson.

Let ABC be the triangle; I the intersection of the perpendiculars AD, BE, CF upon the sides; P any point in the triangle; α, β, γ the perpendiculars from P upon BC, CA, AB. Put AD $= a$, IP $= \rho$, $\angle$ DIP $= \phi$.



Then $\alpha = \frac{1}{3}a - \rho\cos\phi$,

also $\beta = \frac{1}{3}a + \rho\cos(60°-\phi)$,

and $\gamma = \frac{1}{3}a + \rho\cos(60°+\phi)$;

and putting these in $\alpha^2+\gamma^2 > \beta^2$, we have

$$\pm\left\{\rho\cos\phi\cos(60°+\phi) - \tfrac{1}{3}a\cos(60°-\phi)\right\} > \tfrac{1}{3}a\left\{\cos^2(60°-\phi) - \tfrac{1}{2}\cos\phi\cos(60°+\phi)\right\}^{\frac{1}{2}};$$

hence, when the lines α, β, γ form an acute-angled triangle,

we have $$\rho > \tfrac{1}{3}a\,\frac{1+\sqrt{3}\tan\phi + \{3(\tan\phi+\sqrt{3})\tan\phi\}^{\frac{1}{2}}}{1-\sqrt{3}\tan\phi}\sec\phi \quad\text{.........}\ (1),$$

or $$\rho < \tfrac{1}{3}a\,\frac{1+\sqrt{3}\tan\phi - \{3(\tan\phi+\sqrt{3})\tan\phi\}^{\frac{1}{2}}}{1-\sqrt{3}\tan\phi}\sec\phi \quad\text{.........}\ (2),$$

On account of symmetry, we only need take ϕ from 0 to 60°, and multiply by 6. Thus we see that the points indicated by (1) lie entirely outside the triangle; but those in (2) lie wholly within, lying within the area enclosed by the lines ID, IB and the curve line DPG, where IG $= \frac{1}{12}a$. The area of the triangle ABC $= \dfrac{a^2}{\sqrt{3}}$; and putting $=$ for $>$ in (2), we have the chance required

$$= \frac{6\sqrt{3}}{a^2}\int_0^{\frac{1}{6}\pi} \tfrac{1}{2}\rho^2 d\phi$$

$$= 3\log_e(-2) - 2 - \frac{2}{\sqrt{3}}\int_0^{\frac{1}{6}\pi} \frac{(1+\sqrt{3}\tan\phi)(3\tan^2\phi+3\sqrt{3}\tan\phi)^{\frac{1}{2}}\,d\tan\phi}{(1-\sqrt{3}\tan\phi)^2}$$

$$= 3\log_e(-2) - 2 + \frac{108}{\sqrt{3}}\int_\infty^{\sqrt{6}} \frac{(z^2+6)\,z^2dz}{(z^2-12)\cdot(z^2-3)^2}, \text{ putting } \tan\phi = \frac{3\sqrt{3}}{z^2-3},$$

$$= 3\log_e 2 - 2.$$

Again, the condition $\alpha+\gamma<\beta$ gives $\rho\cos(60^\circ-\phi) > \frac{1}{6}a$; whence, taking this as an equality, it represents the line joining D, F; therefore the chance that no triangle can be formed with the three perpendiculars is $\frac{3}{4}$. Consequently, the chance that an obtuse triangle will be formed is

$$1-\tfrac{3}{4}-(3\log_e 2-2) = \tfrac{9}{4}-3\log_e 2.$$

3376. (Proposed by DISCO.)—In the cubic equation $x^3+ax=b$, assume $ax = 3x^2y+3xy^2$, then complete the cube, and solve by a process analogous to the one usually adopted in a quadratic.

Solution by MISS MYRA GREAVES.

Substituting the assumed expression for ax, adding y^3 to both sides, and taking the cube root, we get $\quad x = (y^3+b)^{\frac{1}{3}} - y$ .... (1).

But from the assumption $ax = 3x^2y+3x^2y$ we also get

$$x = \frac{a}{3y} - y \quad \text{.................................... (2).}$$

Equating these two values of x, we have

$$y^6+by^3 = \tfrac{1}{27}a^3 \quad \text{.................................. (3);}$$

and solving this as a quadratic, we get

$$y^3 = -\tfrac{1}{2}b \pm (\tfrac{1}{27}a^3+\tfrac{1}{4}b^2)^{\frac{1}{2}} \quad \text{........................... (4).}$$

Hence y is known from (4); and then, from (2), x may be found, and thus the required solution will be effected.

3358. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—Show that the formulæ, by which the rectangular tangential coordinates of a plane in space are transformed, establish, by mere inspection, the well-known relations between the nine direction-cosines.

Solution by the PROPOSER.

Let two systems of rectangular axes in space, having a common origin O, meet a fixed plane in the points X, Y, Z, X′, Y′, Z′; and let $\frac{1}{OX}, \frac{1}{OY}, \frac{1}{OZ}, \frac{1}{OX'}, \frac{1}{OY'}, \frac{1}{OZ'}$ be put severally equal to ξ, v, ζ, ξ_1, v_1, ζ_1. Let P be the perpendicular from the origin on the fixed plane; then the cosines of the angles which P makes with the two sets of axes are $P\xi$, Pv, $P\zeta$, $P\xi_1$, Pv_1, $P\zeta_1$.

Let the axis OX make with the axes OX′, OY′, OZ′ the angles λ, μ, ν; OY with the same axes the angles λ', μ', ν'; and let OZ make the angles λ'', μ'', ν'', with the same axes: for brevity, put l, m, n, l_1, m_1, n_1, and l_2, m_2, n_2, for the cosines of the above angles. Then, by a common elementary theorem, the cosine of the angle between P and OX is the product of the cosines of the angles, two by two, which the lines P and OX make with the derived axes of the coordinates, namely OX′, OY′, OZ′.

Hence $\quad P\xi = P\xi_1 l + Pv_1 l_1 + P\zeta_1 l_2, \quad$ or $\quad \xi = \xi_1 l + v_1 l_1 + \zeta_1 l_1$;

in like manner, $\quad v = \xi_1 m + v_1 m_1 + \zeta_1 m_2, \quad \zeta = \xi_1 n + v_1 n_1 + \zeta_1 m_2$.

Squaring these expressions, and adding the results; bearing in mind that $\xi^2 + v^2 + \zeta^2 = \xi_1^2 + v_1^2 + \zeta_1^2$, seeing that each expression represents the squared reciprocal of the perpendicular from the common origin on the fixed plane; it follows without any demonstration, on mere inspection alone, that

$$l^2+m^2+n^2 = 1, \quad l_1^2+m_1^2+n_1^2 = 1, \quad l_2^2+m_2^2+n_2^2 = 1,$$

$$ll_1+mm_1+nn_1 = 0, \quad l_1l_2+m_1m_2+n_1n_2 = 0, \quad ll_2+mm_2+nn_2 = 0.$$

Again, if we wish to express the derived coordinates in terms of the original axes, we shall find

$$\xi_1 = \xi l + vm + \zeta n, \quad v_1 = \xi l_1 + vm_1 + \zeta n_1, \quad \zeta_1 = \xi l_2 + vm_2 + \zeta n_2;$$

or, squaring and adding, we shall see on inspection that

$$l^2+l_1^2+l_2^2 = 1, \quad m^2+m_1^2+m_2^2 = 1, \quad n^2+n_1^2+n_2^2 = 1,$$

$$lm+l_1m_1+l_2m_2 = 0, \quad ln+l_1n_1+l_2n_2 = 0, \quad mn+m_1n_1+m_2n_2 = 0.$$

When the axes are translated to a second origin, the *projective* coordinates of the *first* origin on the trihedral planes of the second system of axes parallel to those of the first being p, q, r, we shall find

$$\xi = \frac{\xi_1}{p\xi_1+qv_1+r\zeta_1-1}, \quad v = \frac{v_1}{p\xi_1+qv_1+r\zeta_1-1}, \quad \zeta = \frac{\zeta_1}{p\xi_1+qv_1+r\zeta_1-1}.$$

3184. (Proposed by the Rev. T. P. KIRKMAN, M.A., F.R.S.)—R is any whole number, and $R = m_1e_1 + m_2e_2 + m_3e_3 + \dots$ is every partition in turn of R into positive multiples, one or more, of different numbers $e_1, e_2, e_3 \dots$ Thus $5 = 5.1 = 1.5 = 1.4 + 1.1 = 1.3 + 1.2 = 1.3 + 2.1 =$ &c. are the partitions of $R = 5$. Then

$$\frac{(2R-1)(2R-3)\dots\dots 3.1}{2R.(2R-2)\dots\dots 4.2} = \Sigma \frac{1}{(2e_1)^{m_1}.(2e_2)^{m_2}\dots\dots \Pi m_1.\Pi m_2\dots\dots},$$

where every partition of R adds a term under Σ.

Solution by the PROPOSER.

The proposition to be proved is

$$\left\{1.3.5\ldots\ldots(2\mathrm{R}-1)\right\}^2 = \Pi 2\mathrm{R}\,.\,\Sigma\,\frac{1}{(2e_1)^{m_1}.\,(2e_2)^{m_2}\ldots\ldots\Pi m_1\,.\,\Pi m_2\ldots\ldots}.$$

The number of substitutions $\theta_1, \theta_2, \theta_3 \ldots$ of the second order made with 2R elements, and having no element undisturbed, is $1.3.5\ldots(2\mathrm{R}-1)$. The left side is the number of the products $\theta_1\theta_1, \theta_1\theta_2, \theta_2\theta_1 \ldots\ldots$ The right side is the entire number of substitutions made with 2R elements, and having no circular factor of an odd number of elements; that is, of substitutions having m_1 factors of $2e_1$ elements, m_2 factors of $2e_2$ elements, &c. Let these substitutions be $\phi_1, \phi_2, \phi_3 \ldots\ldots$ We have to prove that the number of products $\theta_a\theta_b$ is that of the substitutions ϕ.

Every substitution $\rho = \theta_a\theta_b$ is positive, for it is made from unity by R transpositions in θ_b followed by R in θ_a, that is, by an even number of them. Also ρ has all its circular factors in pairs, otherwise it could not be the product of θ_a and θ_b, neither of which has an undisturbed element. This is easily seen. Suppose, unity being 1234567890*abcd*,

$$\rho = 2341\ 6785\ 0abcd9$$

of the 12th order. This is the product $\lambda_2\lambda_1$ of the pair

$$\lambda_1 = 1432\ 5876\ 9dcba0, \quad \lambda_2 = 2143\ 6587\ 09dcba,$$

of which λ_1 has three undisturbed elements 1, 5, 9. It is also the product $\mu_2\mu_1$ of the pair

$$\mu_1 = 5876\ 1432\ 9dcba0, \quad \mu_2 = 6587\ 2143\ 09dcba,$$

of which μ_1 has only one element 9 undisturbed.

But it is impossible to express ρ as the product $\theta_a\theta_b$ of two factors of the second order which have both only disturbed elements. Yet if

$$\rho' = 2341\ 6785\ 0a9\ cdb,$$

it is the product $\lambda'_2\lambda'_1$ of the pair

$$\lambda'_1 = 1432\ 5876\ 9a0\ bdc, \quad \lambda'_2 = 2143\ 6587\ 09a\ cbd,$$

of which λ'_1 has four undisturbed elements; and also the product $\mu'_2\mu'_1$ of the pair $\mu'_1 = 5876\ 1432\ bdc\ 9a0, \quad \mu'_2 = 6587\ 2143\ cbd\ 09a,$

of which neither has an element undisturbed. The substitution ρ has not its circular factors in pairs; but ρ' has two of four elements and two of three, and ρ' has consequently, while ρ has not, two factors of the form $\theta_a\theta_b$, made each by transposition of R pairs.

Thus we see, without more formal demonstration, that every substitution $\theta_a\theta_b$ or $\theta_b\theta_a$ is made on some partition

$$2\mathrm{R} = 2m_1\,.\,e_1 + 2m_2\,.\,e_2 + 2m_3\,.\,e_3 + \ldots\ldots,$$

having $2m_1$ circular factors of e_1 elements, $2m_2$ of e_2, &c.; while every substitution of the series $\phi_1, \phi_2, \phi_3 \ldots$ is made in some partition

$$2\mathrm{R} = m_1\,.\,2e_1 + m_2\,.\,2e_2 + m_3\,.\,2e_3 + \ldots\ldots,$$

having circular factors of $2e_1$, $2e_2 \ldots$ elements, doubles of e_1, $e_2 \ldots$, &c.; that

is, $\theta_a \theta_b = \rho$ is always the square of some ϕ. Write down the series of ϕ over ϕ^2 thus,

$$\left.\begin{matrix} \phi_1, \phi_2, \phi_3 \ldots\ldots \\ \phi_1^2, \phi_2^2, \phi_3^2 \ldots\ldots \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}.$$

No two of $\phi_1, \phi_2, \phi_3 \ldots$ can be alike; but ϕ_r^2 may have many square roots $\phi_r, \phi_s, \phi_t \ldots$, and will be written under them all. We must determine under how many of them. The numbers $m_1, m_2 \ldots e_1, e_2 \ldots$ being fixed, the number of different substitutions ϕ given by the partition is

$$\frac{\Pi 2\mathrm{R}}{(2e_1)^{m_1} . (2e_2)^{m_2} \ldots\ldots \Pi m_1 . \Pi m_2 \ldots\ldots} = \mathrm{A};$$

while that of the different substitutions $\theta_a \theta_b = \phi_r^2$ is

$$\frac{\Pi 2\mathrm{R}}{e_1^{2m_1} . e_2^{2m_2} \ldots\ldots \Pi 2m_1 . \Pi 2m_2 \ldots\ldots} = \mathrm{A}';$$

Hence $\mathrm{A} : \mathrm{A}' = e_1^{m_1} . e_2^{m_2} \ldots\ldots \{1.3.5 \ldots (2m_1 - 1)\} \{1.3.5 \ldots (2m_2 - 1)\} \ldots\ldots$ is the number of times that $\mathrm{Q} = \phi_r^2$, made on the first of the above partitions, is found to be the square of a substitution ϕ made in the second partition, giving $\phi_r^2 = \phi_s^2 = \phi_t^2 =$ &c., or the number of times that this ϕ_r will be read in the second line of (B).

Our theorem will be proved if we can show that this Q is found $\mathrm{A} : \mathrm{A}'$ times as $\mathrm{Q} = \theta_a \theta_b$, among the $\{1.3.5 \ldots (2\mathrm{R} - 1)\}^2$ products of that form.

To make this appear, let $2m_1 = 2$, $e_1 = 6$; $2m_2 = 2$, $e_2 = 4$; $2m_3 = 4$, $e_3 = 3$; $2m_4 = 4$, $e_4 = 1$; and let unity of 36 elements be

$$a_1 a_2 a_3 \ldots a_6\, b_1 b_2 b_3 \ldots b_6\, c_1 c_2 c_3 c_4\, d_1 d_2 d_3 d_4\, f_1 f_2 f_3\, g_1 g_2 g_3\, h_1 h_2 h_3\, i_1 i_2 i_3 j_1\, k_1\, l_1\, m_1.$$

One substitution Q of the 12th order, formed in the partition

$$2\mathrm{R} = 36 = 2.6 + 2.4 + 4.3 + 4.1,$$

is $a_6 a_1 a_2 a_3 a_4 a_5\, b_6 b_1 \ldots b_3\, c_4 c_1 c_2 c_3\, d_4 \ldots d_3\, f_3 f_1 f_2\, g_3 g_1 g_2\, h_3 h_1 h_2\, i_3 i_1 i_2 j_1\, k_1\, l_1\, m_1 = \mathrm{Q}$, and this is the product of any consecutive pair of the following:

$$\begin{array}{l}
a_1 a_6 a_5 a_4 a_3 a_2\, b_1 b_6 b_5 b_4 b_3 b_2\, c_1 c_4 c_3 c_2\, d_1 d_4 d_3 d_2\, f_1 f_3 f_2\, g_1 g_3 g_2\, h_1 h_3 h_2\, i_1 i_3 i_2 j_1\, k_1\, l_1\, m_1 = q_1 \\
a_2 a_1 \ldots\ldots a_3\, b_2 b_1 \ldots\ldots b_3\, c_2 c_1 c_4 c_3\, d_2 \ldots d_3\, f_2 f_1 f_3\, g_2 \ldots g_3\, h_2 \ldots h_3\, i_2 \ldots i_3 j_1\, k_1\, l_1\, m_1 = q_2 \\
a_3 \ldots\ldots\ldots a_4\, b_3 \ldots\ldots\ldots b_4\, c_3 \ldots c_4\, d_3 \ldots d_4\, f_3 \ldots f_1\, g_3 \ldots g_1\, h_3 \ldots h_1\, i_3 \ldots i_1 j_1\, k_1\, l_1\, m_1 = q_3 \\
\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = q_4 \\
\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = q_5 \\
a_6 \ldots\ldots\ldots a_1\, b_6 \ldots\ldots\ldots b_1\, c_2 \ldots c_3\, d_2 \ldots d_3\, f_3 \ldots f_1\, g_3 \ldots g_1\, h_3 \ldots h_1\, i_3 \ldots i_1 j_1\, k_1\, l_1\, m_1 = q_6 \\
\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \\
a_6 \ldots\ldots\ldots a_1\, b_6 \ldots\ldots\ldots b_1\, c_4 \ldots c_1\, d_4 \ldots d_1\, f_3 \ldots f_1\, g_3 \ldots g_1\, h_3 \ldots h_1\, i_3 \ldots i_1 j_1\, k_1\, l_1\, m_1 = q_{12}
\end{array}$$

We have
$$\begin{aligned}
q_1 q_2 &= q_2 q_3 = q_3 q_4 = \ldots\ldots = q_{11} q_{12} = q_{12} q_1 = \mathrm{Q}, \\
q_1 q_3 &= q_2 q_4 = q_3 q_5 = \ldots\ldots = q_{11} q_1 = q_{12} q_2 = \mathrm{Q}^2, \\
q_1 q_4 &= q_2 q_5 = q_3 q_6 = \qquad\qquad\qquad = q_{12} q_3 = \mathrm{Q}^3,
\end{aligned}$$

&c. &c.; but none of these products are of the required form $\theta_a \theta_b$, since $q_1, q_2 \ldots$ have all undisturbed elements. The 12 powers of Q, along with the 12, $q_1 q_2 \ldots$, form one of the groups of Theorem G in my Memoir on Groups in the *Memoirs of the Philosophical Society of Manchester*, 1860—61.

By a transposition of pairs, we obtain the required form thus:

$$b_1b_6b_5b_4b_3b_2\ a_1a_6a_5a_4a_3a_2\ d_1d_4d_3d_2\ c_1c_4c_3c_2\ g_1g_3g_2\ f_1f_3f_2\ i_1i_3i_2\ h_1h_3h_2\ k_1j_1m_1l_1 = q'_1$$
$$b_2 \ldots\ldots\ldots b_3\ a_1 \ldots\ldots\ldots a_3\ d_2 \ldots\ldots d_3\ c_2 \ldots\ldots c_3\ g_2 \ldots g_3\ f_2 \ldots f_3\ i_2 \ldots i_3\ h_2h_1h_3\ k_1j_1m_1l_1 = q'_2$$
&c. &c. on to q'_{12},

where none of the 12 q', made by cyclical permutation of subindices in the vertical, has an element undisturbed. We have again the same properties

$$q'_1q'_2 = q'_2q'_3 = \ldots\ldots = q'_{12}q'_1 = \mathrm{Q}, \quad q'_1q'_3 = q'_2q'_4 = \ldots\ldots = q'_{12}q'_2 = \mathrm{Q}^2, \text{ \&c.}$$

The pairs transposed of the $2m_3 = 4$ triplets can be chosen in $1.3\ldots(m_3-1) = 6$ ways; as can also the pairs of the $2m_4 = 4$ undisturbed letters *jklm*. Further, we can cyclically permute by themselves, in the vertical, any pair or pairs with the subindices which they carry. For example, by so permuting the pairs *dc* and *ih*, we have

$$b_1b_6b_5b_4b_3b_2\ a_1a_6a_5a_4a_3a_2\ d_2d_1d_4d_3\ c_2c_1c_4c_3\ g_1g_3g_2\ f_1f_3f_2\ i_2i_1i_3\ h_2h_1h_3\ k_1j_1m_1l_1 = q''_1,$$
$$b_2 \ldots\ldots\ldots b_3\ a_1 \ldots\ldots\ldots a_3\ d_3 \ldots\ldots d_4\ c_2 \ldots\ldots c_4\ g_2g_1g_3\ f_2f_1f_3\ i_3i_2i_1\ h_3h_2h_1\ k_1j_1m_1l_1 = q''_2,$$

which may be continued to q''_{12}. And we have again

$$q''_1q''_2 = q''_2q'' = \ldots\ldots = q''_{12}q''_1 = \mathrm{Q}, \quad q''_1q''_3 = q''_2q''_4 = \ldots\ldots = q''_{12}q''_2 = \mathrm{Q}^2, \text{ \&c.}$$

We have in our power $3^{m_3} = 3^2$ variations of q''_1 by this cyclical permutation of the pairs of triplets, as we can combine three sets of subindices under *gf* with three under *ih*. Hereby it is evident that there are

$$e_1^{m_1}.\, e_2^{m_2} \ldots\ldots \{1.3.5\ldots(2m_1-1)\}\ \{1.3.5\ldots(2m_2-1)\} \ldots\ldots = \mathrm{A} : \mathrm{A}'$$

different pairs $\theta_a\theta_b$, each giving $\theta_a\theta_b = \phi_r^2$, ϕ_r^2 being any substitution which has $2m_1$ circular factors of e_1 elements, $2m_2$ of e_2, $2m_3$ of e_3, &c., just as there are $6.4.3^2.1^2.1.1.3.3$ ways of writing the substitution Q in the required form. Thus our theorem is demonstrated.

The above investigation is the main part of the development withheld for want of space in Art. 77 of my Memoir.

3328. (Proposed by B. WILLIAMSON, M.A.)—A cone of the second degree passes through a fixed plane section of a given quadric; if its second plane of intersection always passes through a given point, find the locus of the vertex of the cone.

I. *Solution by* Rev. R. TOWNSEND, M.A., F.R.S.

Denoting by S the given quadric, by M and P the given plane and its pole with respect to S, and by Q and N the given point and its polar plane with respect to S, then, from the known property that the vertices of the two cones which pass through any two plane sections of any quadric surface are collinear with and harmonically conjugate to each other with respect to those of the two which envelope the quadric along the two

sections, it appears at once, by the method of homographic division applied to any plane section of it passing through the line PQ, that the required locus is a second quadric S′, passing at once through the two conics of intersection of the original quadric S with the two planes M and N, and having the point P and the plane N pole and polar to each other with respect to itself; particulars which, of course, determine it completely.

The reciprocal property—that, if the vertex of a variable cone enveloping a fixed quadric move on a fixed plane, its two planes of intersection with any fixed cone circumscribed to the surface envelope a second quadric, inscribed at once to the fixed cone and to that which envelopes the original quadric along its section by the fixed plane, and having the vertex of the latter and the plane of contact of the former pole and polar to each other with respect to itself—may be proved precisely in the same manner.

II. *Solution by* J. J. Walker, M.A.

Let the quadric surface be $S=0$, the fixed plane section $ax+by+cz+dw$ or $P=0$, and the other plane of intersection, which always passes through the point $(x'y'z'w')$, $Q=0$; and let P', Q' be the results of substituting x', y', z', w' for x, y, z, w respectively in P, Q.

The equation to any quadric surface passing through the sections of S by P and Q will be of the form $\Sigma \equiv \lambda S + PQ = 0$, and this will be a cone, having $(xyzw)$ as its vertex, if simultaneously $\dfrac{d\Sigma}{dx}=0, \;\ldots\ldots\; \dfrac{d\Sigma}{dw}=0$. Combining with these the equation $Q'=0$, we have a system of five equations among which to eliminate linearly λ and the four coefficients in Q. The result is, where S_x stands for $\dfrac{dS}{dx}$,

$$0 = \begin{vmatrix} P+ax, & ay, & az, & aw, & S_x \\ bx, & P+by, & bz, & bw, & S_y \\ cx, & cy, & P+cz, & cw, & S_z \\ dx, & dy, & dz, & P+dw, & S_w \\ x', & y', & z', & w', & 0 \end{vmatrix}$$

$$= P^2 \left\{ P'S - P\left(x'S_x + y'S_y + z'S_z + w'S_w\right) \right\}.$$

Rejecting the irrelevant factor P^2, the other factor evidently represents a quadric surface intersecting S in the sections made by the plane P and by the polar plane of the point $(x'y'z'w')$.

3161. (Proposed by R. W. Genese, B.A.)—ABC, A′B′C′ are two triangles in perspective; P, Q, R points on their axis of homology. If AP, BQ, CR cointersect, so will A′P, B′Q, C′R.

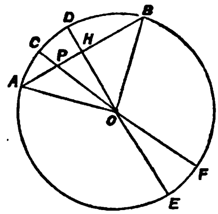
Solution by W. H. Laverty, B.A.; *the* Proposer; *and others.*

This is a projective property; therefore project the figure so that the axis of homology goes to infinity. Then any two lines, as AP, A′P, be-

come parallel, and the triangles become similar and similarly situated figures. And it is evident that if three lines be drawn from A, B, C, meeting in a point, three parallels to these through A′, B′, C′ respectively will also be concurrent.

3015. (Proposed by A. Martin.)—A point is taken at random in the surface of a circle, and a chord drawn through it; find (1) the average length of the chord, and (2) the average area of the segment cut off by the chord.

Solution by the Proposer.



1. Let P be any point in the circle, and AB a chord drawn through it. Draw the diameter COF through P, and another DHOE at right angles to AB.

Let r be the radius of the circle, OP $= x$, the angle OPB $= \phi$, and M_1, M_2 the mean values required in (1) and (2) respectively.

Then $\qquad OH = x \sin\phi,$

$$AB = 2AH = 2HB = 2\,(r^2 - x^2\sin^2\phi)^{\frac{1}{2}},$$

and

$$M_1 = \int_0^{\frac{1}{2}\pi}\int_0^r 2\,(r^2-x^2\sin^2\phi)^{\frac{1}{2}} \times 2\pi x\,dx\,d\phi \div \int_0^{\frac{1}{2}\pi}\int_0^r 2\pi x\,dx\,d\phi$$

$$= \frac{8}{\pi r^2}\int_0^{\frac{1}{2}\pi}\int_0^r (r^2-x^2\sin^2\phi)^{\frac{1}{2}}\,x\,dx\,d\phi = \frac{8r}{3\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{1-\cos^3\phi}{\sin^2\phi}\right)d\phi$$

$$= \frac{8r}{3\pi}\int_0^{\frac{1}{2}\pi}\left(\cos\phi + \frac{1}{1+\cos\phi}\right)d\phi = \frac{8r}{3\pi}\Big[\sin\phi + \tan\tfrac{1}{2}\phi\Big]_0^{\frac{1}{2}\pi} = \frac{16r}{3\pi}.$$

2. The segment ADBH $= r^2\cos^{-1}\left(\dfrac{x\sin\phi}{r}\right) - x\sin\phi\,(r^2-x^2\sin^2\phi)^{\frac{1}{2}}$;

$$\therefore\ M_2 = \frac{4}{\pi r^2}\int_0^{\frac{1}{2}\pi}\int_0^r \left\{r^2\cos^{-1}\left(\frac{x\sin\phi}{r}\right) - x\sin\phi\,(r^2-x^2\sin^2\phi)^{\frac{1}{2}}\right\}x\,dx\,d\phi$$

$$= \frac{r^2}{2\pi}\int_0^{\frac{1}{2}\pi}\left(2\pi - 4\phi + \frac{\phi}{\sin^2\phi} - \frac{\cos\phi}{\sin\phi} - 2\sin\phi\cos\phi\right)d\phi$$

$$= \frac{r^2}{4}\left(\pi - \frac{2}{\pi}\right).$$

3201. (Proposed by J. J. Walker, M.A.)—From any point tangents are drawn to a cissoid, and circles are described passing through the three

points of contact and through the three points in which the tangents again meet the cissoid. Prove that the pencils formed by joining the cusp with the pairs of four points in which the two circles meet the cissoid are homographic.

Solution by the PROPOSER.

Writing the equation to the cissoid $x^3-y^2z=0$, where y and z are the distances of any point from the tangent at cusp (O) and asymptote respectively, x from a line through the cusp perpendicular to the tangent; and $x'y'z'$ being the point (P) from which tangents PT_1, PT_2, PT_3 are drawn to the cissoid, these meeting the curve again in the points t_1, t_2, t_3, let the circle through T_1, T_2, T_3 meet the cissoid in a fourth point T_0, and that through t_1, t_2, t_3 in a fourth point t_0; then it may be shown (see *Proceedings of the Mathematical Society of London*, No. 18) that the equations to the pencils $O(T_0, T_1, T_2, T_3)$, $O(t_0, t_1, t_2, t_3)$ are

$$3x'z'y^4+4y'z'y^3x-18x'^2x^2y^2+8y'^2x^4=0,$$

and
$$12x'z'y^4-4y'z'y^3x-9x'^2x^2y^2+y'^2x^4=0.$$

But the I of these equations will be found to be of the same value in each case, viz., $48x'y'^2z'+27x'^4$; and, similarly, the J of each is equal to $-144x'^3y'^2z'-8y'^4z'^2+27x'^6$; and the equality of these invariants is the condition of the two pencils being homographic (*Quarterly Journal*, Vol. X., p. 56).

3329. (Proposed by Professor CAYLEY.)—It is required to show that every permutation of 12345 can be produced by means of the cyclical substitution (12345), and the interchange (12).

I. *Solution by the* Rev. ROBERT HARLEY, F.R.S.

If any one of the five numbers, say the last, be fixed, and the other four be permuted *inter se*, there will result 24 arrangements; and if upon each of these repeated operations be performed by the cyclical substitution (12345), every permutation of the five numbers will be produced. But every permutation of the first four numbers can be formed by means of the interchanges (12), (13), (14), (23), (24), (34). The question therefore is reduced to this, to show that the last five of these interchanges can be effected by means of the first and the cyclical substitution (12345).

Now write, for shortness, $S=(12345)$ and $i=(12)$; then we have

$$(13)=iS^2iSiS,\qquad (23)=SiS^4,\qquad (34)=S^2iS^3,$$
$$(14)=S^3iS^2iSiS^3,\qquad (24)=SiS^2iSi,$$

and the theorem is established.

It is worthy of remark that the above expressions in S and i have a variety of equivalents. Thus, to take the first and third,

$$iS^2iSiS=iSiS^4i,\quad\text{and}\quad SiS^4=S^2iSiSi.$$

For, since $S^5=1$, $i^2=1$, and $(Si)^4=1$, we have

$$(Si)^3Si=1=i^2;\ \text{therefore}\ (Si)^3S=i=iS^5;\ \text{therefore}\ (Si)^3=iS^4;$$

and $\qquad iS^2iSiS = iS(Si)^2S = iS(Si)^3i = iSiS^4i;$

also $\qquad SiS^4 = S(Si)^3 = S^2iSiSi.$

It may also be remarked that

$$(15) = (iS)^3S = S^4iS, \qquad (35) = S^4iS(iS^3)^2,$$
$$(25) = (Si)^2S^2i = iS^4iSi, \qquad (45) = S^3iS^2.$$

II. *Solution by the* Proposer.

It is sufficient to show that the interchanges (13), (14), (15) can be so produced; for then, with the interchanges (12), (13), (14), (15), we can, by at most two such interchanges, bring any number into any place.

Now, writing $P = (12345)$, $\alpha = (12)$, we have

$$\begin{aligned}(12) &= \alpha,\\ (13) &= \alpha P \alpha P^4 \alpha,\\ (14) &= \alpha P \alpha P^4 \alpha P^2 \alpha P^3 \alpha P \alpha P^4 \alpha,\\ (15) &= P^4 \alpha P,\end{aligned}$$

as can be at once verified; and the theorem is thus proved.

I remark that, starting with any two or more substitutions, and combining them in every possible manner (each of them being repeatable an indefinite number of times), we obtain a "group"; viz., this is either (as in the foregoing example) the system of all the substitutions (or say the entire group), or else it is a system the number of whose terms is a submultiple of the whole number of substitutions. The interesting question is, to determine those two or more substitutions which, by their combination as above, do *not* give the entire group; for in this way we should arrive at all the forms of a submultiple group.

3263. (Proposed by S. Roberts, M.A.)—A straight line meets a curve of the nth degree in the points $O_1, O_2, \dots O_n$. Let U_{n-1} be the first polar of the point at infinity through which the straight line passes, and let V_{n-1} be another curve of the $(n-1)$th degree which passes through the same points at infinity as the first polar U_{n-1}. Through $O_1, O_2, \dots O_n$ draw a set of vectors $v_1, v_2, \dots v_n$, and suppose that v_1 meets V_{n-1} in the points $S_1, S_2, \dots S_{n-1}$, and U_{n-1} in the points $T_1, T_2, \dots T_{n-1}$; that v_2 meets V_{n-1} in the points $S'_1, S'_2, \dots S'_{n-1}$, and U_{n-1} in the points T'_1, T'_2, T'_{n-1}, and so on. Then will

$$\Sigma \frac{OS_1 . OS_2 \dots OS_{n-1}}{OT_1 . OT_2 \dots OT_{n-1}} = n,$$

the summation being extended to all the vectors.

Solution by the Proposer.

Jacobi has given the following theorem (see Serret's *Cours d'Algèbre Supérieure*, t. I., p. 626).—Let $\phi(xy)$ be a rational and integer func-

tion of a degree inferior to $m+n-2$, and let $f_m(xy)$, $F_n(xy)$ be rational and integer functions of the degrees m and n respectively. Then

$$\Sigma \frac{\phi(\alpha\beta)}{\frac{df_m(\alpha\beta)}{d\alpha}\frac{dF_n(\alpha\beta)}{d\beta} - \frac{df_m(\alpha\beta)}{d\beta}\frac{dF_n(\alpha\beta)}{d\alpha}} = 0,$$

the summation extending to all the solutions α, β which satisfy the system

$$f_m(xy) = 0, \quad F_n(xy) = 0.$$

If $f_m(xy)$ is of the form $bx - ay + c$, so that, equated to cipher, it represents a straight line; then

$$\frac{d}{dx}f_m(xy)\frac{d}{dy}F_n(xy) - \frac{d}{dy}f_m(xy)\frac{d}{dx}F_n(xy)$$
$$= a\frac{d}{dx}F_n(xy) + b\frac{d}{dy}F_n(xy) = 0$$

represents the first polar of the point where $f_m(xy) = 0$ meets the line at infinity.

The summation extending to all the intersections of line $f_m(xy) = 0$ and $F_n(xy) = 0$, we have, as a particular consequence of Jacobi's theorem,

$$\Sigma \frac{a\frac{d}{d\alpha}F_n(\alpha\beta) + b\frac{d}{d\beta}F_n(\alpha\beta) + \phi_{n-2}(\alpha\beta)}{a\frac{d}{d\alpha}F_n(\alpha\beta) + b\frac{d}{d\beta}F_n(\alpha\beta)} = n.$$

But following the construction of the question, and taking α', β' as the coordinates of O_1, we have also

$$\frac{a\frac{d}{d\alpha'}F_n(\alpha'\beta') + b\frac{d}{d\beta'}F_n(\alpha'\beta') + \phi_{n-2}(\alpha'\beta')}{a\frac{d}{d\alpha'}F_n(\alpha'\beta') + b\frac{d}{d\beta'}F_n(\alpha'\beta')} = \frac{O_1S_1 \,.\, O_1S_2 \ldots O_1S_{n-1}}{O_1T_1 \,.\, O_1T_2 \ldots O_1T_{n-1}},$$

as can be seen by transferring the origin to the point $O_1(\alpha', \beta')$, and substituting polar for Cartesian coordinates. Hence we have the result in the question.

2401. (Proposed by MORGAN JENKINS, M.A.)—Show that

$$\Delta^n 0^{n+r} = [(\Sigma_1^n n)^r (1)] \underline{|n};$$

that is, the rth operation of $\Sigma_1^n n$ upon 1, not of Σ_1^n upon n; and that

$$(\Sigma_1^n n)^r (1) = n^{[r+1]} \phi_r(n),$$

where $n^{[r+1]}$ denotes $n(n+1)\ldots(n+r)$, and $\phi_r(n)$ is an algebraic function of n of the $(r-1)$th degree with integral or fractional coefficients.

Solution by the PROPOSER.

From the first equation we have

$$\Delta^n 0^{n+1} = \frac{n(n+1)}{1.2} . \lfloor n ;$$

$$\Delta^n 0^{n+2} = \left\{ \Sigma_1^n \frac{n^2(n+1)}{1.2} \right\} \lfloor n = \frac{1}{2} \left\{ \Sigma_1^n (n^3+n^2) \right\} \lfloor n$$

$$= \frac{1}{2} \left\{ \left(\frac{n(n+1)}{1.2} \right)^2 + \frac{n(n+1)(2n+1)}{1.2.3} \right\} \lfloor n$$

$$= \frac{n(n+1)(n+2)}{1.2.3} . \frac{(3n+1)}{4} . \lfloor n ,$$

$$\Delta^n 0^{n+3} = \left(\frac{n(n+1)}{1.2} \right)^2 . \frac{(n+2)(n+3)}{3.4} \lfloor n ,$$

$$\Delta^n 0^{n+4} = \frac{15n(n+1)^2 - 2(5n+1)}{48} . \frac{n^{[5]}}{\lfloor 5} . \lfloor n ,$$

$$\Delta^n x^{n+r} = \left\{ \Delta_1 + (1+\Delta_1)\Delta_2 \right\}^n x^{n+r-1} . x$$

(where Δ_1, Δ_2 apply to x^{n+r-1} and x respectively)

$$= (\Delta_1^n x^{n+r-1})\, x + n . \Delta_1^{n-1} (1+\Delta_1)\, x^{n+r-1} . \Delta_2 x + \ldots\ldots$$

If $x=0$, the first term, and all terms which contain $\Delta_2^2 x$, &c., vanish;

therefore $\qquad \Delta^n 0^{n+r} = n \Delta^{n-1} 0^{n+r-1} + n \Delta^n 0^{n+r-1}.$

Suppose the law proved for $\Delta^n 0^{n+r-1}$: then, putting $\Delta^n 0^{n+r} = u_n$, we

have $\qquad u_n = n u_{n-1} + n \left[(\Sigma_1^n n)^{r-1} (1) \right] \lfloor n .$

Hence, if u_{n-1} contain $\lfloor n-1$, u_n will contain $\lfloor n$. But u_1 does contain 1, because $\Delta_1 0^m = 1$ for all values of m. Hence u_n contains $\lfloor n$.

Let $\qquad u_n = v_n \lfloor n , \quad u_{n-1} = v_{n-1} \lfloor n-1 ;$

then $\qquad v_n = v_{n-1} + n (\Sigma_1^n n)^{r-1} (1), \quad \Delta v_{n-1} = n (\Sigma_1^n n)^{r-1} (1).$

Hence $\qquad v_n = C + (1+\Delta^{-1}) . n (\Sigma_1^n n)^{r-1} (1).$

But since $\Delta^{-1} u_n$ denotes summation exclusive of u_n, $(1+\Delta^{-1})$ denotes summation inclusive of u_n, that is $[(1+\Delta^{-1})\, n] = (\Sigma_1^n n)$;

therefore $\qquad v_n = C + (\Sigma_1^n n)^r (1).$

But when $n=1$, $v_n = 1$; and it is evident that $(\Sigma_1^n n)^r (1)$ must be equal to 1 when $n=1$. Therefore $C=0$, and the formula will hold for $\Delta^n . 0^{n+r}$ if it hold for $\Delta^n 0^{n+r-1}$. But $\Delta^n 0^n = \lfloor n$; hence, by induction, the formula holds for $\Delta^n 0^{n+r}$.

Again, assume that $\quad (\Sigma_1^n n)^{r-1} (1) = n^{[r]} \phi_{r-1}(n),$

where $\phi_{r-1}(n)$ is an algebraic function of n with integral indices and

rational coefficients. Then $(\Sigma_1^n n)^{r-1}(1)$ may be put in the form

$$A_r n^{[r]} + A_{r+1} n^{[r+1]} + \ldots\ldots$$

Now, since $A_r n \,.\, n^{[r]} + A_{r+1} n \,.\, n^{[r+1]} + \ldots\ldots$ may be put in the form

$$B_r n^{[r]} + B_{r+1} n^{[r+1]} + \ldots\ldots,$$

and $$(1+\Delta^{-1})\, n^{[r]} \text{ or } D\Delta^{-1} n^{[r]} = (r+1)^{-1} n^{[r+1]};$$

therefore $(\Sigma_1^n n)^r(1)$ will contain $n^{[r+1]}$ if $(\Sigma_1^n n)^{r-1}(1)$ contain $n^{[r]}$. Hence the law holds, since $(\Sigma_1^n n)(1) = \dfrac{n^{[2]}}{\underline{|2}}$.

It is evident that $\phi_r(n)$ must be of the $(r-1)$th degree; for every operation $\Sigma_1^n n$ raises the power of n by 2, one by the multiplication of n, another by the summation. Hence $(\Sigma_1^n n)^r(1)$ must be of the $2r$-th degree, since $\Sigma_1^n n$ is of the 2nd, and since $n^{[r+1]}$ is of the $(r+1)$th, $\phi_r n$ must be of the $(r-1)$th degree.

3180. (Proposed by W. H. H. Hudson, M.A.)—From a point O on a cubic four tangents are drawn to touch the cubic at A, B, C, D. If O be given by the equations $x=y=z$, and D by the equations $lx=my=nz$, while $x=0$, $y=0$, $z=0$ represent the lines BC, CA, AB, the equation to the cubic will be $lx^2(y-z)+my^2(z-x)+nz^2(x-y)=0$,
and the four tangents will form a harmonic pencil if l, m, n be in harmonic progression.

Solution by the Rev. J. Wolstenholme, M.A.

The given conditions are nine in number, one for O and two for each of the points A, B, C, D. Hence they are sufficient to determine a cubic (unless they happen to be the nine common points of two cubics, which is not the case here). But the given equation

$$lx^2(y-z) + my^2(z-x) + nz^2(x-y) = 0$$

passes through the point $x=y=z$, or O; it touches the line $y=z$ at the point $y=0$, $z=0$, *i.e.*, touches OA at A, and similarly OB, OC at B, C; also it passes through the point $lx=my=nz$ or D; and the equation of the polar conic of the point $x=y=z$ is

$$lx(y-z) + my(z-x) + nz(x-y) = 0,$$

showing that OD touches the cubic at D.

The equations of the four tangents are

$$y=z,\quad z=x,\quad x=y,\quad \text{and}\quad x\left(\frac{1}{m}-\frac{1}{n}\right) + y\left(\frac{1}{n}-\frac{1}{l}\right) + z\left(\frac{1}{l}-\frac{1}{m}\right) = 0,$$

or $$(y-x)+(x-z)=0,\ \ (x-z)=0,\ \ (y-x)=0,$$
$$\left(\frac{1}{n}-\frac{1}{l}\right)(y-x)-\left(\frac{1}{l}-\frac{1}{m}\right)(x-z)=0;$$

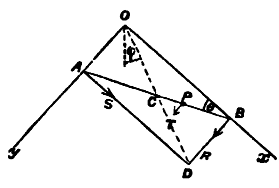
whence one of the anharmonic ratios is $\left(\frac{1}{n}-\frac{1}{l}\right)\div\left(\frac{1}{m}-\frac{1}{l}\right)$,

which will be harmonic if $\frac{2}{l}=\frac{1}{m}+\frac{1}{n}$.

[As there is some ambiguity in the notation of pencils, it may be desirable to mention that by the range (ABCD) we understand $\frac{\text{AB}\,.\,\text{CD}}{\text{AC}\,.\,\text{BD}}$, sign being always attended to. It follows that the anharmonic ratio of the pencil $u=\mu_1 v$, $u=\mu_2 v$, $u=\mu_3 v$, $u=\mu_4 v$, taken in this order, is
$$\frac{(\mu_1-\mu_2)(\mu_3-\mu_4)}{(\mu_1-\mu_3)(\mu_2-\mu_4)}.]$$

3109. (Proposed by the Rev. F. D. THOMSON, M.A.)—A rod is attached by small, smooth, weightless rings at its extremities to two fixed rods forming a right angle, with the angle uppermost. The system is placed in a vertical plane with the fixed rods equally inclined to the horizon, and the moveable rod is in the position of equilibrium. A small, smooth, heavy ring is placed near the centre of the moveable rod, and the rod is then slightly displaced. Determine the motion, when an oscillation is possible.

Solution by the PROPOSER.



1. Let AB $=2a$ be the sliding rod, C its middle point, P the heavy ring, CP $=r$, θ the angle which AB makes with the fixed rod OB. And let R, S, T be the pressures at B, A, P.

Let $\theta=45^\circ-\phi$, then by supposition
$$r,\ \phi,\ \frac{d\theta}{dt},\ \frac{dr}{dt},\ \frac{d^2\theta}{dt^2},\ \frac{d^2r}{dt^2},$$
are small quantities of the first order.

2. Consider the accelerations of P along and perpendicular to the rod AB.

Ac. of P along CP = ac. relative to C + ac. of C in the same direction
$$=\frac{d^2r}{dt^2}+a\frac{d^2\phi}{dt^2},$$
neglecting small quantities of the second order;

therefore $$g\cos(\theta+45^\circ)=g\sin\phi=\frac{d^2r}{dt^2}+a\frac{d^2\phi}{dt^2};$$

or $$\frac{d^2r}{dt^2}+a\frac{d^2\phi}{dt^2}=g\phi\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

acc. of P perp. to CP = acc. of C in the same direction (to the 1st order of small quantities)
= 0 (to the same degree of approximation);

therefore $$\mathrm{T} = mg \quad\text{.......................................(2)}.$$

Also, taking moments about the instantaneous centre of rotation D, we have, for the motion of the rod, the equation

$$\tfrac{4}{3}\mathrm{M}a^2\frac{d^2\theta}{dt^2} = \mathrm{T}(a\cos 2\theta - r) + \mathrm{M}ga\sin\phi,$$

or $$-\tfrac{4}{3}\mathrm{M}a^2\frac{d^2\phi}{dt^2} = \mathrm{T}(2a\phi - r) + \mathrm{M}ga\phi \quad\text{............................(3)};$$

therefore, eliminating T,

$$-\tfrac{4}{3}\mathrm{M}a^2\frac{d^2\phi}{dt^2} = mg(2a\phi - r) + \mathrm{M}ga\phi \quad\text{...........................(4)}.$$

Assume for the solutions of (1) and (4)

$$\phi = \alpha\cos(nt+\mathrm{B}), \quad r = c\cos(nt+\mathrm{B});$$

Then, substituting, $$\left.\begin{array}{l} -cn^2 - a\alpha n^2 = g\alpha \\ \tfrac{4}{3}\mathrm{M}a^2\alpha n^2 = mg(2a\alpha - c) + \mathrm{M}ga\alpha \end{array}\right\};$$

therefore, eliminating c and α,

$$\tfrac{4}{3}\mathrm{M}a^2n^4 - (3m+\mathrm{M})\,agn^2 - mg^2 = 0 \quad\text{.........................(5)},$$

a quadratic which gives one positive value of n^2.

If α and c be the initial values of ϕ and r, B=0, and the motion is given by $$\left.\begin{array}{l} \phi = \alpha\cos nt \\ r = c\cos nt \end{array}\right\}.$$

It appears, from (5), that the time of oscillation decreases as the relative mass of the ring increases.

3. It is assumed in the above investigation that an oscillation is *possible*, and that therefore r is always small.

But this implies that after the displacement, the centre of gravity of the system is *higher* than before. This gives the condition

$$a\cos\alpha + \frac{m}{m+\mathrm{M}}\,c\sin\alpha < a;$$

or, since α is small, $$c < \frac{m+\mathrm{M}}{2m}\,a\alpha,$$

which gives a necessary limit to the initial distance of the ring from the centre of the rod.

3218. (Proposed by the Rev. J. Blissard.)—Prove that

$$1^6 - 2^6 + 3^6 - \ldots \pm x^6 = \tfrac{1}{2}(-1)^{x+1}(x^6 + 3x^5 - 5x^3 + 3x) \;\text{.........}\; (1);$$

$$(1.2.3)^2 - (2.3.4)^2 + \ldots \pm \left\{x(x+1)(x+2)\right\}^2 \ldots\ldots$$

$$\ldots\ldots \tfrac{1}{2}(-1)^{x+1}(x^6 + 9x^5 + 28x^4 + 33x^3 + 7x^2 - 6x) \;\text{......}\; (2).$$

I. *Solution by the* Rev. J. Wolstenholme, M.A.

$$\Delta\left\{\tfrac{1}{2}(-1)^{x-1}(x^6+3x^5-5x^3+3x)\right\}$$
$$=\tfrac{1}{2}(-1)^x\left\{(x+1)^6+3(x+1)^5-5(x+1)^3+3(x+1)+x^6+3x^5-5x^3+3x\right\}$$
$$=\tfrac{1}{2}(-1)^x(2x^6+12x^5+30x^4+40x^3+30x^2+12x+2)=(-1)^x(x+1)^6,$$

therefore $\Sigma(-1)^x(x+1)^6=C+\tfrac{1}{2}(-1)^{x-1}(x^6+3x^5-5x^3+3x)$;

and putting $x=1$, $\quad 1=C+1$ or $C=0$,

therefore $1^6-2^6+3^6-\ldots+(-1)^{x-1}x^6=\tfrac{1}{2}(-1)^{x-1}(x^6+3x^5-5x^3+3x)$.

So $\quad \Delta\left\{\tfrac{1}{2}(-1)^{x-1}(x^6+9x^5+28x^4+33x^3+7x^2-6x)\right\}$

$$=\tfrac{1}{2}(-1)^{x-1}\left\{x^6+9x^5+28x^4+33x^3+7x^2-6x+(x+1)^6+9(x+1)^5\right.$$
$$\left.+28(x+1)^4+33(x+1)^3+7(x+1)^2-6(x+1)\right\}$$
$$=\tfrac{1}{2}(-1)^{x-1}\left\{2x^6+24x^5+116x^4+288x^3+326x^2+264x+72\right\}$$
$$=(-1)^{x-1}(x^3+6x^2+11x+6)^2=(-1)^{x-1}\left\{(x+1)(x+2)(x+3)\right\}^2,$$

therefore $\tfrac{1}{2}(-1)^{x-1}(x^6+9x^5+28x^4+33x^3+7x^2-6x)$

$$=C+\Sigma(-1)^{x-1}\left\{(x+1)(x+2)(x+3)\right\}^2;$$

and putting $x=1$, $\quad 36=(1.2.3)^2+C$ or $C=0$,

therefore $(1.2.3)^2-(2.3.4)^2+\ldots+(-1)^{x-1}\left\{x(x+1)(x+2)\right\}^2$

$$=\tfrac{1}{2}(-1)^{x-1}(x^6+9x^5+28x^4+33x^3+7x^2-6x).$$

II. *Solution by* Stephen Watson.

When x is even, writing $\lfloor n+m$ for $n(n+1)\ldots(n+m)$, we have

$1^6-2^6+3^6-\ldots\ldots-x^6=$ sum to n terms of the series whose general term is

$$\left\{(2n-1)^6-(2n)^6\right\}=\Sigma\left\{(2n+1)^6-2^6(n+1)^6\right\}$$
$$=\Sigma\left\{-192\lfloor n+4+1200\lfloor n+3-160\lfloor n+2+300\lfloor n+1-64n-63\right\}$$
$$=(n-1)\left\{-32\lfloor n+4+240\lfloor n+3-40\lfloor n+2+100\lfloor n+1-32n\right\}-63n$$
$$=-(32n^6+48n^5-20n^3+3n),\text{ which, since } 2n=x, \text{ becomes}$$
$$=-\tfrac{1}{2}(x^6+3x^5-5x^3+3x)\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

By a like process, when x is odd, we have

$$1^6-2^6+3^6-\ldots+x^6=\tfrac{1}{2}(x^6+3x^5-5x^3+3x)\ \ldots\ldots\ldots\ldots\text{(B)}.$$

Again, when x is even,

$$(1.2.3)^2-(2.3.4)^2+\ldots\ldots-\left\{x(x+1)(x+2)\right\}^2$$

$=$ sum to n terms of the series whose general term is

$$\left\{(2n-1)\,2n\,(2n+1)\right\}^2-\left\{2n(2n+1)(2n+2)\right\}^2$$
$$=-12\Sigma(n+1)^2(2n+3)^2(4n+5)$$
$$=(n-1)\left\{-32\lfloor n+4+144\lfloor n+3-144\lfloor n+2-84\lfloor n+1\right\}-270n(n+1)$$
$$=-(32n^6+144n^5+224n^4+132n^3+14n^2-6n),\text{ which, since } 2n=x,$$
$$=-\tfrac{1}{2}(x^6+9x^5+28x^4+33x^3+7x^2-6x)\ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}.$$

Similarly, when x is odd,

$$(1\,.\,2\,.\,3)^2-(2\,.\,3\,.\,4)^2+\ldots\ldots+\{x(x+1)(x+1)\}^2$$
$$=\tfrac{1}{2}(x^6+9x^5+28x^4+33x^3+7x^2-6x)\;\ldots\ldots\ldots\ldots\ldots\ldots(\text{D}).$$

Hence (A) and (B) prove (1), and (C) and (D) prove (2).

3233. (Proposed by S. Roberts, M.A.)—Show that the number of the conditions which must be satisfied, in order that a curve of the nth degree may be symmetrical about an axis is $\frac{1}{4}(n^2+2n-8)$ if n be even, and $\frac{1}{4}(n^2+2n-7)$ if n be odd.

Solution by the Proposer.

Suppose that the axis is given, the axis of x, for instance; the equation of the curve will be of the form $F(y^2, x) = 0$.

We have therefore to make the coefficients of terms, whose arguments contain odd powers of y, vanish.

If n be even, the number of such coefficients is $\frac{1}{4}n(n+2)$.

If n be odd, ,, ,, $\frac{1}{4}(n+1)^2$.

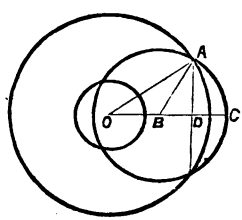
The fixing of the axis is equivalent to two conditions.

We have therefore, for the numbers required,

$$\tfrac{1}{4}n(n+2)-2=\tfrac{1}{4}(n^2+2n-8)\quad\text{and}\quad\tfrac{1}{4}(n+1)^2-2=\tfrac{1}{4}(n^2+2n-7).$$

3316. (Proposed by A. Martin.)—A point is taken at random in the surface of a given circle, and a straight line of given length drawn from it in a random direction. Find the chance that the line will meet the circumference of the circle.

I. *Solution by the* Proposer.



Put $OA = r$, $AB = a =$ the given line, and let $OB = x$. Then $BD = \dfrac{r^2-a^2-x^2}{2x}$, and the angle $ABC = \cos^{-1}\left(\dfrac{r^2-a^2-x^2}{2ax}\right)$, and while the point is at B the chance of meeting the circumference is $\dfrac{1}{\pi}\cos^{-1}\left(\dfrac{r^2-a^2-x^2}{2ax}\right)$.

Hence, if p be the chance required, we have

$$p = \int_{r-a}^{r} \cos^{-1}\left(\frac{r^2-a^2-x^2}{2ax}\right) 2\pi x\,dx \div \pi \int_0^r 2\pi x\,dx$$

$$= \frac{2}{r^2\pi} \int_{r-a}^{r} x \cos^{-1}\left(\frac{r^2-a^2-x^2}{2ax}\right) dx$$

$$= \frac{1}{\pi r^2} x^2 \cos^{-1}\left(\frac{r^2-a^2-x^2}{2ax}\right) - \frac{1}{\pi}\cos^{-1}\left(\frac{r^2+a^2-x^2}{2ar}\right)$$

$$+ \frac{1}{2\pi r^2}\left\{4a^2r^2 - (r^2+a^2-x^2)^2\right\}^{\frac{1}{2}} \quad \text{(between limits)}$$

$$= \frac{2}{\pi}\sin^{-1}\left(\frac{a}{2r}\right) + \frac{a\,(4r^2-a^2)^{\frac{1}{2}}}{2\pi r^2}.$$

If $a = r$, then $p = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}$, [and when $a = 2r$, the chance is 1, as it ought to be].

II. *Solution by* S. Watson; Rev. J. L. Kitchin, M.A.; *and others.*

Let B be the point, BQ the line, O the centre of the given circle, OC a radius through B, and BA = BQ meeting the circumference in A. Put $OC = r$, $OB = x$, $BQ = a$, $\angle OBQ = \phi$. The total number of positions of B and BQ is $r^2\pi \times 2\pi = 2r^2\pi^2$; hence the required chance is

$$p = \frac{1}{2r^2\pi^2} \int_{r-a}^{r} 2(\pi - \phi)\, 2\pi x\,dx$$

$$= 1 - \frac{(r-a)^2}{r^2} - \frac{2}{\pi r^2}\int_{r-a}^{r} x\,dx \cos^{-1}\frac{x^2+a^2-r^2}{2ax};$$

whence, integrating 'by parts,' this becomes

$$= 1 - \frac{1}{\pi}\cos^{-1}\left(\frac{a}{2r}\right) - \frac{1}{\pi r^2}\int_{r-a}^{r} \frac{(x^2+r^2-a^2)\,x\,dx}{\left\{4a^2x^2 - (x^2+a^2-r^2)^2\right\}^{\frac{1}{2}}}$$

$$= 1 - \frac{1}{\pi}\cos^{-1}\left(\frac{a}{2r}\right) + \frac{1}{2\pi r^2}\int_{2ra}^{a^2} \frac{(2r^2 - X)\,dX}{(4r^2a^2 - X^2)^{\frac{1}{2}}} \quad (\text{where } X = r^2 + a^2 - x^2)$$

$$= \frac{2}{\pi}\sin^{-1}\left(\frac{a}{2r}\right) + \frac{a}{2\pi r^2}(4r^2 - a^2)^{\frac{1}{2}}.$$

The above is on the supposition that r is greater than a, but the result is the same when a lies between r and $2r$.

3284. (Proposed by the Editor.)—Integrate the differential equation

$$(x^2+y^2)\frac{d^2y}{dx^2} = n\left(x\frac{dy}{dx} - y\right)\left(1 + \frac{dy^2}{dx^2}\right).$$

I. *Solution by the* Rev. J. WOLSTENHOLME, M.A.; Rev. J. L. KITCHIN, M.A.; J. J. WALKER, M.A.; *and others.*

Here we have

$$\frac{\frac{d^2y}{dx^2}}{1+\frac{dy^2}{dx^2}} = n\,\frac{x\frac{dy}{dx}-y}{x^2+y^2};\quad \text{therefore}\quad \tan^{-1}\left(\frac{dy}{dx}\right) = n\tan^{-1}\left(\frac{y}{x}\right)+a.$$

Let $y = x\tan\theta$, then $\tan\theta + \dfrac{x}{\cos^2\theta}\dfrac{d\theta}{dx} = \tan(n\theta+a)$,

whence $$x\frac{d\theta}{dx} = \frac{\cos\theta\sin\left\{(n-1)\theta+a\right\}}{\cos(n\theta+a)},$$

or $$\frac{1}{x}\frac{dx}{d\theta} = \cot\left\{(n-1)\theta+a\right\}-\tan\theta;$$

therefore $$\log\frac{x}{a} = \frac{1}{n-1}\log\left\{\sin(n-1)\theta+a\right\}+\log\cos\theta,$$

or $$x^{n-1} = a^{n-1}\sin\left\{(n-1)\theta+a\right\}\cos^{n-1}\theta,$$

or $$(x^2+y^2)^{\frac{1}{2}(n-1)} = a^{n-1}\sin\left\{(n-1)\tan^{-1}\frac{y}{x}+a\right\},$$

the complete integral, which form, however, fails when $n=1$.

The interpretation of the equation is obviously that the chord of curvature through the pole is to the radius vector as $2:n$, and the general solution is $$r^{n-1} = a^{n-1}\sin\left\{(n-1)\theta+a\right\};$$
but when $n=1$, the solution is the equiangular spiral $r = b\epsilon^{m\theta}$.

II. *Solution by* W. ROBERTS, Jun.; Rev. J. L. KITCHIN, M.A.; *and others.*

The given equation may be put into the form

$$1 = n\,.\,\frac{x\frac{dy}{dx}-y}{x^2+y^2}\,.\,\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\,.\,\frac{dx}{ds}\;\ldots\ldots\ldots\ldots\ldots\ldots\;(1).$$

Changing to polar coordinates (r,θ), putting ϕ for the angle between the tangent and radius-vector, and integrating, we have

$$\theta+\phi = n\theta + \text{a constant}\;\ldots\ldots\ldots\ldots\ldots\ldots\;(2);$$

therefore $\phi = (n-1)(\theta+\beta)$ say, and $\tan\phi = r\dfrac{d\theta}{dr} = \tan(n-1)(\theta+\beta)$.

Integrating, we have $r^{n-1} = a^{n-1}\sin(n-1)(\theta+\beta)\;\ldots\ldots\ldots\ldots\ldots\;(3)$,
which agrees with the first solution, a being $=(n-1)\beta$.

Putting $n=2$ in (3), and returning to rectangular coordinates, we get
$$x^2+y^2 = a(x\sin\beta+y\cos\beta),\ \text{the equation to a circle.}$$

When $n=1$, the solution is evidently, from (2), $r = b\epsilon^{m\theta}$, the equation to a logarithmic spiral.

III. *Solution by* J. J. WALKER, M.A.

Let $\frac{dy}{dx}=p,\ \frac{d^2y}{dx^2}=q,\ y=ux,\ q=\frac{v}{x}$. Then

$$\frac{dx}{x}=\frac{du}{p-u},\ \text{and}\ vdu=(p-u)\,dp \ \ldots\ldots\ldots\ldots (1, 2).$$

The given equation becomes $(1+u^2)\,v=n\,(p-u)\,(1+p^2)$,

combining which with (2), $\frac{du}{1+u^2}=\frac{ndp}{1+p^2}$.

Integrating, when $n=1$, we have

$\tan^{-1}u+\tan^{-1}c=\tan^{-1}p$, or $p=\frac{c+u}{1-cu}$; whence $p-u=\frac{c\,(1+u^2)}{1-cu}$.

From this and (1), $\frac{dx}{x}=\frac{du}{c\,(1+u^2)}-\frac{n\,du}{1+cu^2}$,

the integral of which is easily found to be

$$\tan^{-1}\frac{y}{x}=c\log\,(x^2+y^2)+c'.$$

[For another solution of the case where $n=2$, see the Solution of Quest. 2617, *Reprint*, Vol. XII., p. 103.]

3379. (Proposed by Sir JAMES COCKLE, F.R.S.)—Given, that α and β are functions of x only, and that

$$f(\alpha)=\frac{\alpha}{2}\left(\frac{d^2\alpha}{dx^2}+\frac{d\beta}{dx}\right)-\frac{1}{4}\left(\frac{d\alpha}{dx}+\beta\right)\left(\frac{d\alpha}{dx}-\beta\right)\ \ldots\ldots\ldots\ldots (1);$$

find a perfect solution of the linear partial biordinal

$$\alpha^2r-t=\beta q+\gamma z \ \ldots\ldots\ldots\ldots (2)$$

for the cases in which $\gamma=f(\alpha)$ and in which $\gamma=f(-\alpha)$. Also, discuss one or more particular instances in which the result, or the process of solution, may be modified.

Solution by the PROPOSER.

Monge's system for (2) is $dz=pdx+qdy \ \ldots\ldots\ldots\ldots (3)$,

$$\alpha^2dy^2-dx^2=(\alpha dy+dx)\,(\alpha dy-dx)=0 \ \ldots\ldots\ldots\ldots (4),$$

$$\alpha^2dpdy-dqdx-(\beta q+\gamma z)\,dxdy=0 \ \ldots\ldots\ldots\ldots (5).$$

In pursuance of the rule which I gave in my Solution of Question 3205, make q a constant. Then, on reduction, (5) becomes

$$dp=\frac{1}{\alpha^2}\,(\beta q+\gamma z)\,dx \ \ldots\ldots\ldots\ldots (6).$$

From (4) select $\alpha dy=dx \ \ldots\ldots\ldots\ldots (7)$,

then, if v be some function of x only, we have

$$vdp=\frac{v}{\alpha^2}\,(\beta q+\gamma z)\,dx,\quad \frac{dv}{dx}\,pdx=\frac{dv}{dx}\left(dz-\frac{q}{\alpha}\,dx\right)\ \ldots\ldots (8, 9).$$

Adding (8) and (9), we obtain

$$d(vp) = \frac{dv}{dx}dz + \frac{\gamma v}{a^2}zdx + \left(\frac{\beta v}{a^2} - \frac{1}{a}\frac{dv}{dx}\right)qdx$$

$$= d\left(\frac{dv}{dx}z\right) + \left(\frac{\gamma v}{a^2} - \frac{d^2v}{dx^2}\right)zdx + \left(\frac{\beta v}{a^2} - \frac{1}{a}\frac{dv}{dx}\right)qdx$$

$$= d\left(\frac{dv}{dx}z\right) + q\left(\frac{\beta v}{a^2} - \frac{1}{a}\frac{dv}{dx}\right)dx \quad \text{............................... (10),}$$

provided always that $$\frac{\gamma v}{a^2} - \frac{d^2v}{dx^2} = 0 \quad \text{................................ (11).}$$

Integrating the last member of (10) as if q were constant, we have

$$vp = q\int\left(\frac{\beta v}{a^2} - \frac{1}{a}\frac{dv}{dx}\right)dx + \frac{dv}{dx}z \quad \text{............... (12).}$$

Now we have employed in the integration the form $dy = \frac{dx}{a}$. The other form is $dy = -\frac{dx}{a}$. Hence, if we put (12) under the form $u = 0$, we have the condition $$\frac{du}{dq} = -\frac{1}{a}\frac{du}{dp} \quad \text{................................. (13),}$$

which is equivalent to $$\int\left(\frac{\beta v}{a^2} - \frac{1}{a}\frac{dv}{dx}\right)dx = \frac{v}{a} \quad \text{...................... (14),}$$

whence $$\frac{\beta v}{a^2} - \frac{1}{a}\frac{dv}{dx} = \frac{1}{a}\frac{dv}{dx} - \frac{v}{a^2}\frac{da}{dx} \quad \text{..................... (15);}$$

or, reducing, $$\frac{1}{2a}\left(\frac{da}{dx} + \beta\right) = \frac{1}{v}\frac{dv}{dx} \quad \text{........................... (16).}$$

But (11) must be satisfied. Differentiating (16), we find

$$\frac{1}{2a}\left(\frac{d^2a}{dx^2} + \frac{d\beta}{dx}\right) - \frac{1}{2a^2}\left(\frac{da}{dx} + \beta\right)\frac{da}{dx} = \frac{1}{v}\frac{d^2v}{dx^2} - \left(\frac{1}{v}\frac{dv}{dx}\right)^2 \quad \text{...... (17);}$$

and, by means of (11) and (16), eliminating v from (17), there results

$$\frac{1}{2a}\left(\frac{d^2a}{dx^2} + \frac{d\beta}{dx}\right) - \frac{1}{2a^2}\left(\frac{da}{dx} + \beta\right)\frac{da}{dx} = \frac{\gamma}{a^2} - \frac{1}{4a^2}\left(\frac{da}{dx} + \beta\right)^2,$$

or, $$\gamma = \frac{a}{2}\left(\frac{d^2a}{dx^2} + \frac{d\beta}{dx}\right) - \frac{1}{4}\left(\frac{da}{dx} + \beta\right)\left(\frac{da}{dx} - \beta\right) \quad \text{........ ... (18),}$$

the dexter of which is the same as that of (1). Hence, when $\gamma = f(a)$, the equation (12) is a first integral of (2). Eliminating the sign of integration from (12) by means of (14), dividing the result by v, and then eliminating v by (16), we deduce from (12), by combining its integral with that of (7), $$p - \frac{q}{a} - \left(\frac{da}{dx} + \beta\right)\frac{z}{2a} = \frac{1}{v}\phi\left(y - \int\frac{dx}{a}\right) \quad \text{............ (19),}$$

wherein the function ϕ is arbitrary, and the value of v, deduced from (16), is $$v = e^{\int\left(\frac{da}{dx} + \beta\right)\frac{dx}{2a}} = \sqrt{a}\, e^{\int\frac{\beta}{2a}dx} \quad \text{..................... (20).}$$

For the integration of (19), we have

$$dx = -ady = \frac{vdz}{\frac{dv}{dx}z + \phi\left(k - 2\int\frac{dx}{a}\right)}, \quad \text{where } k = y + \int\frac{dx}{a} \quad \text{.... (21).}$$

Consequently $z = v\chi\left(y + \int \frac{dx}{a}\right) + v\int \phi\left(k - 2\int \frac{dx}{a}\right)\frac{dx}{v^2}$ ...... (22),

wherein, after integration, for k is to be substituted its value as given by (21). This is the perfect solution of (2) for the case of $\gamma = f(a)$. It contains one explicit arbitrary function χ, and an implicit one $\int \frac{\phi}{v^2}\,dx$.

When β is constant $\int \frac{\phi}{v^2}\,dx = \int e^{-\int \frac{\beta}{a}dx}\frac{\phi}{a}\,dx$........................ (23).

Hence, if we put generally $\int^n \phi(m)\,dm^n = \phi_n$,

and integrate (23) by parts, we have

$$\int e^{-\int \frac{\beta}{a}dx}\frac{\phi}{a}\,dx = -e^{-\int \frac{\beta}{a}dx}\frac{\phi_1}{2} - \int e^{-\int \frac{\beta}{a}dx}\frac{\phi_1}{a}\frac{\beta}{2}\,dx = \&c. = \&c.$$

$$= e^{-\int \frac{\beta}{a}dx}\,\Sigma(-1)^n\,\frac{\beta^{n-1}\phi_n}{2^n}\begin{bmatrix}\text{from } n = 1\\ \text{to } \quad n = a\end{bmatrix} \;\text{............}\; (24).$$

But the last member of (24) is of the form

$$e^{-\int \frac{\beta}{a}dx}\phi\left(k - 2\int \frac{dx}{a}\right).$$

Hence, when β is constant, the perfect solution (22) becomes

$$z = \sqrt{a}\,e^{-\int \frac{\beta}{a}dx}\chi\left(y + \int \frac{dx}{a}\right) + \sqrt{a}\,e^{-\int \frac{\beta}{a}dx}\phi\left(y - \int \frac{dx}{a}\right) \;\text{......}\; (25).$$

The present confirm certain results which I have already given in the *Reprint* (see Vol. X. pp. 81—84). The Mongian factor $ady + dx$ yields no solution when $\gamma = f(a)$. But when $\gamma = f(-a)$, we obtain a first integral by changing the sign of a in (19). If β be constant, then $f(a) = f(-a)$, and both (19) and the equation which arises from it by changing the sign of a are first integrals, and the perfect solution contains two explicit arbitrary functions. If $\frac{da}{dx} + \beta = 0$ .... (26),

then $f(a)$ vanishes and (6) gives $dp + \frac{q}{a^2}\frac{da}{dx}\,dx = 0$,

and, integrating as if q were constant, we find

$$p - \frac{q}{a} = \phi\left(y - \int \frac{dx}{a}\right) \;\text{........................}\; (27),$$

as a first integral, for the requirement of the rule is fulfilled. The same result would be obtained by substitution in (19). Boole has stated (*Diff. Eq.*, p. 366) that, when a given equation admits of a first integral at all, it will admit of two such, excepting the case in which the sinister of the equation corresponding to (4) of this solution is a complete square. In the sentence preceding, Boole merely stated that either factor of that sinister *may* lead to a final integral system. In a subsequent page (p. 373, Ex. 4) he adopts one system, remarking that there is another, not integrable in

the required form. He observes (pp. 368, 369) that the two first integrals stand in a certain conjugate relation. The present solution shows that conjugate forms may be, not conjugate integrals of the same equation, but integrals of conjugate equations.

3356. (Proposed by Professor CAYLEY.)—If the roots $(\alpha, \beta, \gamma, \delta)$ of the equation $(a, b, c, d, e)(u, 1)^4 = 0$ are no two of them equal; and if there exist unequal magnitudes θ, ϕ such that

$$(\theta+\alpha)^4 : (\theta+\beta)^4 : (\theta+\gamma)^4 : (\theta+\delta)^4 = (\phi+\alpha)^4 : (\phi+\beta)^4 : (\phi+\gamma)^4 : (\phi+\delta)^4;$$

show that the cubinvariant $ace-ad^2-b^2e-c^3+2bcd = 0$; and find the values of θ, ϕ.

Solution by J. J. WALKER, M.A.

Let k be the common ratio; then $\alpha, \beta, \gamma, \delta$ satisfy the equation

$$(1-k)u^4+4(\theta-k\phi)u^3+6(\theta^2-k\phi^2)u^2+4(\theta^3-k\phi^3)u+\theta^4-k\phi^4 = 0;$$

consequently
$$\frac{1-k}{a} = \frac{\theta-k\phi}{b} = \frac{\theta^2-k\phi^2}{c}.$$

Eliminating k, the result is divisible by $\theta-\phi$, and gives

$$a\theta\phi-b(\theta+\phi)+c = 0 \quad\text{.............................(1).}$$

Also
$$\frac{\theta-k\phi}{b} = \frac{\theta^2-k\phi^2}{c} = \frac{\theta^3-k\phi^3}{d};$$

and again eliminating k, and dividing by $(\theta-\phi)\theta$, the result is

$$b\theta\phi-c(\theta+\phi)+d = 0 \quad\text{.............................(2).}$$

Finally, from
$$\frac{\theta^2-k\phi^2}{c} = \frac{\theta^3-k\phi^3}{d} = \frac{\theta^4-k\phi^4}{e},$$

$$c\theta\phi-d(\theta+\phi)+e = 0 \quad\text{.............................(3).}$$

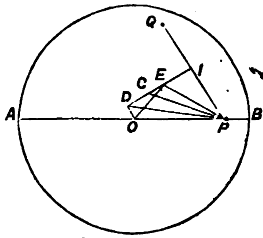
Eliminating $\theta\phi$ and $-(\theta+\phi)$ linearly between (1), (2), (3), there results

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = 0.$$

Solving any two of (1), (2), (3)—say (1), (2)—for $\theta\phi$, $\theta+\phi$, there result, to determine θ, ϕ,
$$\theta\phi = \frac{bd-c^2}{ac-b^2}, \quad \theta+\phi = \frac{ad-bc}{ac-b^2}.$$

3335. (Proposed by the EDITOR.)—Two points are taken at random inside a given circle; show that the average area of all the circles that can be drawn through these two points so as to touch the given circle, is two-fifths of the area of that circle.

Solution by STEPHEN WATSON.



Let O be the centre of the given circle; P and Q any two points within it; AB a diameter through P; and D, E the centres of circles through P and Q tangential to the given circle. Join OD, OE, PD, PE; and draw also DE, producing it, if necessary, to bisect PQ in I. Put $OB = a$, $PQ = ay$, $PD = r$, $PE = r_1$, and $\angle OPQ = \phi$. Then we have

$$(a-r)^2 = OD^2$$
$$= a^2x^2 + r^2 - 2arx\cos(\phi - DPI)$$
$$= a^2x^2 + r^2 - a^2xy\cos\phi - 2ax\sin\phi\,(r^2 - \tfrac{1}{4}a^2y^2)^{\frac{1}{2}};$$

therefore $4X^2r^2 - 4a(m^2 + xy\cos\phi)r + a^2(m^4 + 2m^2xy\cos\phi + x^2y^2) = 0 \ldots(1)$,

where $X^2 = 1 - x^2\sin^2\phi$, and $m^2 = 1 - x^2$.

The roots of the equation (1) are r, r_1; hence we have

$$r + r_1 = \frac{a(m^2 + xy\cos\phi)}{X^2}, \quad \text{and} \quad rr_1 = \frac{a^2(m^4 + 2m^2xy\cos\phi + x^2y^2)}{4X^2} \ldots(2);$$

therefore the sum of the areas of the circles of radii r, r_1 is

$$\pi(r^2 + r_1^2) = \left\{\frac{a^2(m^2 + xy\cos\phi)^2}{X^4} - \frac{a^2(m^4 + 2m^2xy\cos\phi + x^2y^2)}{2X^2}\right\}\pi \ldots(3).$$

An element of the circle at $P = 2a^2\pi x\,dx$, and at $Q = a^2y\,dy\,d\phi$. The limits of y are from 0 to $X + x\cos\phi$, then, putting $\pi - \phi$ for ϕ, and adding the results, to include the case of Q below AB; the limits of ϕ are from 0 to $\frac{1}{2}\pi$, and the result doubled; and of x from 0 to 1. Also, the whole number of tangential circles is $2a^4\pi^2$; hence the required average is

$$\frac{1}{2a^4\pi^2}\int_0^1 2a^2\pi x\,dx \int_0^{\frac{1}{2}\pi} 2d\phi \int_0^{X + x\cos\phi} (3)\, a^2y\,dy,$$

which, being integrated for y between these limits, then $\pi - \phi$ put for ϕ, and the results added, becomes

$$= a^2\int_0^1 x\,dx \int_0^{\frac{1}{2}\pi} d\phi \left\{\tfrac{4}{3}(3 - m^2)X^2 - \tfrac{2}{3}(2 - m^2)m^2 - \frac{(5 - m^2)m^4}{6X^2} - \frac{m^6}{3X^4}\right\}$$

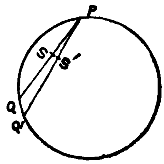
$$= \pi a^2 \int_0^1 x\,dx\,(1 - \tfrac{1}{2}m^3) = \tfrac{3}{5}\pi a^2.$$

[It will be observed that the average area of the circles is the same fractional part of the given circle as that which, in Question 1843 (*Reprint*, Vol. XIII., p. 95), has been found to express the chance that a circle drawn through three random points on the surface of the given circle will lie wholly within it.]

3382. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If a curve similar to $r^m = a^m\cos m\theta$, or an equiangular spiral, be described having

contact of the third order with a given curve V, and S be the pole of this curve, the tangent at S to the locus of S, and at P to the curve V, make equal angles with PS.

Solution by J. F. Moulton, B.A.



These curves are all such that the chord of curvature through the pole bears a constant ratio to the radius vector; and two consecutive curves will have a contact of the second order with each other, or will have a common circle of curvature. Hence, if P be the point of osculation, PSQ, PS′Q′ chords of curvature through the poles of two consecutive curves, SS′ will be parallel to QQ′; or, taking the limit, the tangent at S to the locus of S is parallel to the tangent at Q to the circle of curvature, or makes with PS the same angle as is made by the tangent at P.

This includes the parabola (focus pole), rectangular hyperbola (centre pole), cardioid (cusp pole), lemniscate (node pole). The cases $n = \pm 1$ are excluded, as we cannot generally draw a circle or a straight line having contact of the third order.

If we have to draw a curve similar to a given curve, we have obviously four *degrees of freedom*, viz., the position of any point (the pole), the direction of any straight line, and the length of any straight line. If the curve have to be similar and similarly situated, one of these is confiscated. Also, in the case of the circle or equiangular spiral, which is independent of direction, one degree of freedom is lost; but by varying the angle of the spiral we have four parameters, so that we can always draw an equiangular spiral having contact of the third order with a given curve at any point.

3391. (Proposed by the Editor.)—Show that the values of θ which satisfy the equation $\tan\theta = 2\cos 2\theta$ are given by

$$\tan\theta = \tfrac{1}{3}\left\{(28+\sqrt{783})^{\frac{1}{3}} + (28-\sqrt{783})^{\frac{1}{3}} - 2\right\} = \cdot 695620769.$$

Solution by Miss Myra Greaves.

The given equation is equivalent to

$$\tan\theta = 2(\cos^2\theta - \sin^2\theta) = 2\cos^2\theta\,(1-\tan^2\theta) = 2\,\frac{1-\tan^2\theta}{1+\tan^2\theta},$$

which reduces to $\quad \tan^3\theta + 2\tan^2\theta + \tan\theta - 2 = 0\,.$ .... (1).

Assuming $\tan\theta = x - \frac{2}{3}$, and reducing, we get $\quad x^3 - \frac{1}{3}x = \frac{56}{27}$ ....(2).

By making $a = -\frac{1}{3}$, $b = \frac{56}{27}$, in the expression for y^3 in my solution of Question 3376, and substituting in the first expression for x in the same solution, we finally get the required expression for $\tan\theta$, whether we take the upper or lower sign in the expression for y^3. And $\tan\theta$ being thus found, the required value of θ is easily obtained from the usual tables.

Note *by the* Proposer.

If, in (1), we put $\tan\theta = x + \dfrac{1}{9x} - \dfrac{2}{3}$, the equation becomes

$$x^6 - \tfrac{56}{27}x^3 = -\tfrac{1}{729}, \quad \text{whence} \quad x = \tfrac{1}{3}(28 \pm \sqrt{783})^{\frac{1}{3}},$$

and $$\tan\theta = \tfrac{1}{3}\left\{(28+\sqrt{783})^{\frac{1}{3}} + (28-\sqrt{783}) \; -2\right\}$$
$$= \tfrac{1}{3}(3{\cdot}8254555 + {\cdot}2614070 - 2) = {\cdot}6956208.$$

Tan θ has the value given in the question, so that $\theta = n\pi + 34^\circ\ 49'\ 23{\cdot}7''$. But the readiest method of finding the *numerical* value of tan θ is by Horner's method, observing that (1) has two imaginary roots, and a real root between ·6 and ·7.

The process is as follows:—

1 | 2	+1	−2 (·695620769
6	156	1536
26	256	−464...
6	192	434709
32	448..	−29291...
6	3501	26043375
38 9	48301	−3247625
9	3582	3138915 6
39 8	51883 . .	−108709 4
9	203 7 5	104681 0
40 75	52086 7 5	−4028 4
5	204 0 0	3663 9
40 80	52290 7 5	−364 5
5	24 5 1	314 0
40 85	52315 2 6	50 5
	24 5 1	
	52339 7 7	
	8	
	52340 5	
	8	
	52341 3	

3385. (Proposed by A. Martin.)—A point is taken at random inside a given square or rectangle, and a random line drawn through it; find the probability that the line intersects opposite sides of the figure.

I. *Solution by* Hugh M'Coll.

1. Let P be a random point inside a *square*, and x, y its coordinates referred to two sides of the square, which intersect at the point O (say). Let M, N be the points at which the random line intersects the lines $x=0$, $y=0$ respectively; and let $OM = m$, $ON = n$. Let $PNQ = \theta$, in which Q denotes the point whose coordinates are $y=0$, $x=-\infty$. Let q denote the probability that the random line cuts off the square a triangle whose vertex is O; then $4q$ = the probability that *some* triangle is cut off with *some* corner of the

square as vertex, and the probability required in the question $= 1-4q$. Taking the side of the square $= a$, a random point in the square, with a random line through it, will be equivalent to random values of x and y between 0 and a, combined with a random value of θ between 0 and π; and q denotes the probability that these random values of the three variables will satisfy the four conditions $m < a$, $m > 0$, $n < a$, $n > 0$. Since $m = y + x \tan\theta$, and $n = x + y\cot\theta$, it is easily seen that these four conditions are equivalent to the three conditions $\theta < \frac{1}{2}\pi$, $y < a - x\tan\theta$, $y < (a-x)\tan\theta$. The second of these conditions *includes the third* when $a - x\tan\theta < (a-x)\tan\theta$, that is to say, when $\theta > \frac{1}{4}\pi$; and *is included in the third* when $\theta < \frac{1}{4}\pi$. Hence, when y alone is taken at random, and

$\theta < \frac{1}{4}\pi$, we get $\quad q = \frac{1}{a}\left\{(a-x)\tan\theta\right\} = \left(1 - \frac{x}{a}\right)\tan\theta,$

the average value of which, as x passes from 0 to a, is

$$\frac{1}{a}\int_0^a \left(1 - \frac{x}{a}\right)\tan\theta\, px = \tfrac{1}{2}\tan\theta,$$

for in this case dx may be put for px. In like manner, when $\theta > \frac{1}{4}\pi$ but $< \frac{1}{2}\pi$, and y and x successively taken at random, we get

$$q = \frac{1}{a}\int_0^a \frac{1}{a}(a - x\tan\theta)\, px = \frac{1}{a}\int_0^{a\cot\theta} \frac{1}{a}(a - x\tan\theta)\, dx = \tfrac{1}{2}\cot\theta;$$

and when $\theta > \frac{1}{2}\pi$, we get $q = 0$. Hence, finally, when y, x, and θ are all three taken at random, we get

$$q = \frac{1}{\pi}\left\{\int_{\frac{1}{2}\pi}^{\pi} 0\, d\theta + \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \tfrac{1}{2}\cot\theta\, p\theta + \int_0^{\frac{1}{4}\pi} \tfrac{1}{2}\tan\theta\, p\theta\right\}$$

$$= \frac{1}{\pi}\left\{\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \tfrac{1}{2}\cot\theta\, d\theta + \int_0^{\frac{1}{4}\pi} \tfrac{1}{2}\tan\theta\, d\theta\right\}$$

$$= \frac{1}{\pi}\int_0^{\frac{1}{4}\pi} \tan\theta\, d\theta, \quad \text{for} \int_\beta^\alpha \cot\theta\, d\theta = \int_{\frac{1}{2}\pi-\alpha}^{\frac{1}{2}\pi-\beta} \tan\theta\, d\theta,$$

$$= \frac{1}{\pi}(\log_e \sec\tfrac{1}{4}\pi - \log_e 1) = \frac{1}{2\pi}\log_e 2;$$

and therefore $\qquad 1 - 4q = 1 - \frac{2}{\pi}\log_e 2.$

2. Next, take the case of a *rectangle*. For shortness' sake I will give the following definitions:—

Def. 1. $a > b > c > \ldots$ denotes that a, b, c, &c. are in descending order of magnitude; and $a < b < c < \ldots$, denotes that a, b, c, &c. are in an ascending order.

Def. 2. If q denotes the probability of a certain event depending on the values of certain variables x, y, &c.; then $q(y)$ denotes this probability when y *alone* is taken at random, the other variables being considered fixed or given; $q(y, x)$ denotes the probability when y *and* x *alone* are taken at random; and so on.

Let q, x, y, θ be interpreted as in Art. 1, the *longer* side, say, being taken in this case as the axis of x. Let length : breadth $= a : 1$; and, no linear unit being given, let us (for simplicity) take the *breadth* as our unit. As

in Art. 1 it may be shown that the conditions to be satisfied by the random point and line are $\frac{1}{2}\pi > \theta > 0$, $y < 1 - x\tan\theta$, $y < (a-x)\tan\theta$; and also that these three conditions are equivalent to the two conditions $\frac{1}{2}\pi > \theta > \cot^{-1} a$, $y < 1 - x\tan\theta$, or else to the two conditions $\cot^{-1} a > \theta > 0$, $y < (a-x)\tan\theta$.

Given $\frac{1}{2}\pi > \theta > \cot^{-1} a$; then

$$q(y, x) = \frac{1}{a}\int_0^a (1 - x\tan\theta)\, px = \frac{\cot\theta}{2a}.$$

Given $\cot^{-1} a > \theta > 0$; then

$$q(y, x) = \frac{1}{a}\int_0^a (a-x)\tan\theta\, px = \tfrac{1}{2}a\tan\theta.$$

Given $\pi > \theta > \frac{1}{2}\pi$; then $\quad q(y, x) = 0$.

Therefore

$$q(y, x, \theta) = \frac{1}{\pi}\left\{\int_{\frac{1}{2}\pi}^{\pi} 0\, d\theta + \int_{\cot^{-1} a}^{\frac{1}{2}\pi} \frac{\cot\theta}{2a}\, p\theta + \int_0^{\cot^{-1} a} \tfrac{1}{2}a\tan\theta\, p\theta\right\}$$

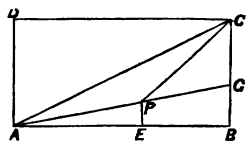
$$= \frac{1}{4\pi a}(1 + a^2)\log_e(1 + a^2) - \frac{a}{2\pi}\log_e a.$$

And the required probability

$$= 1 - 4q(y, x, \theta) = 1 + \frac{2a}{\pi}\log_e a - \frac{1}{\pi a}(1 + a^2)\log_e(1 + a^2).$$

[For the meanings of the symbols px, $p\theta$, see Mr. M'Coll's article on *Probability Notation*, on p. 20 of this volume of the *Reprint*.]

II. *Solution by* Stephen Watson.



Take the case of a rectangle ABCD, and in the triangle ACB take any point P. Join CP and AP, producing the latter to meet BC in G, and draw PE perpendicular to AB. Put $AB = a$, $BC = b$, $AC = c$, $AE = x$, $\angle CAB = \alpha$, $\angle PAB = \phi$. The number of lines that can be drawn through P to cut the adjacent sides AB, BC is $= \angle CPG = CAP + ACP$, and the number, so far as the angle CAP is concerned, is

$$\int_0^a dx \int_0^{\alpha} (\alpha - \phi)\, x d(\tan\phi) = -\tfrac{1}{2}a^2\log\cos\alpha = \tfrac{1}{2}a^2\log\frac{c}{a} \;\ldots\ldots\; (1).$$

Similarly, the number in respect to the angle ACP is $\frac{1}{2}b^2\log\frac{c}{b}$; hence multiplying by 4, because any of the 4 pairs of adjacent sides may be cut by the line, and dividing by πab, the total number of lines that can be drawn through all points in the rectangle, we have the chance of cutting

adjacent sides $\quad = \dfrac{2}{\pi}\left(\dfrac{a}{b}\log\dfrac{c}{a} + \dfrac{b}{a}\log\dfrac{c}{b}\right) = \dfrac{2}{\pi}\log\dfrac{c^{\frac{c^2}{ab}}}{a^{\frac{a}{b}}\, b^{\frac{b}{a}}},$

and the chance of cutting *opposite* sides is $1 - \frac{2}{\pi} \log \frac{c^{\frac{c^2}{ab}}}{a^{\frac{a}{b}} b^{\frac{b}{a}}}$.

When $a=b$, these become respectively $\frac{2}{\pi} \log 2$ and $1 - \frac{2}{\pi} \log 2$.

3397. (Proposed by G. O'Hanlon.)—Find three parabolas such that a tangent of one will be cut harmonically by the other two.

Solution by the Rev. R. Townsend, M.A., F.R.S.

The envelope of a variable line cut harmonically by any two fixed conics being the conic which touches the eight tangents to the two at their four points of intersection (see Salmon's *Conic Sections*, 5th ed., Art. 335); for two parabolas whose axes are parallel the envelope conic is consequently a third parabola; and therefore, &c.

3357. (Proposed by C. W. Merrifield, F.R.S.)—If four points lie in a plane, the locus of the centres of all conics which pass through them has for its centre the centre of gravity of the four points, and the asymptotes of the locus are diameters of the parabolas of the system.

Solution by F. D. Thomson, M.A.

It may be shown, as in Salmon's *Conics*, p. 143, that the equation to a conic making intercepts λ, λ', μ, μ' on the coordinate axes is

$$\mu\mu'x^2 + 2hxy + \lambda\lambda'y^2 - \mu\mu'(\lambda+\lambda')x - \lambda\lambda'(\mu+\mu')y + \lambda\lambda'\mu\mu' = 0,$$

which is a parabola if $h = \pm\sqrt{(\lambda\lambda'\mu\mu')}$.

Also the locus of the centres of the system of conics obtained by varying h is
$$2\mu\mu'x^2 - 2\lambda\lambda'y^2 - \mu\mu'(\lambda+\lambda')x + \lambda\lambda'(\mu+\mu')y = 0,$$
the coordinates of the centre of which are $\frac{1}{4}(\lambda+\lambda')$, $\frac{1}{4}(\mu+\mu')$, or the centre of gravity of the four fixed points.

The asymptotes of the locus are parallel to $\mu\mu'x^2 - \lambda\lambda'y^2 = 0$, whereas diameters of the two parabolas are parallel to

$$\sqrt{(\mu\mu')}\,x + \sqrt{(\lambda\lambda')}\,y = 0, \quad \sqrt{(\mu\mu')}\,x - \sqrt{(\lambda\lambda')}\,y = 0,$$

which proves the proposition.

3349. (Proposed by R. TUCKER, M.A.)—Given focus, further directrix, and major axis of an ellipse; give a simple construction for a pair of conjugate diameters without drawing the ellipse.

Solution by the PROPOSER.

Referring to the figure accompanying my Solution of Quest. 2880 (*Reprint*, Vol. XII., p. 109), and remarking that the orthogonal circle is not there drawn, my construction is as follows:—Draw HO perpendicular to the directrix; and with centre H and radius equal to major axis, describe a circle. Then with centre O describe a circle orthogonal to the former: this cuts HO in S, the nearer focus; hence the centre C is known. Join, as in Quest. 2880, S and H with any point R on the tangent X′R to the orthogonal circle, perpendicular to HO on the side remote from H; then lines through C, parallel to HR and perpendicular to RS respectively, give the directions of conjugate diameters. If with centre C we describe a circle orthogonal to circle (O), its radius will be the distance between the extremities of the major and minor axes. The conjugate diameters can now be determined.

3387. (Proposed by S. ROBERTS, M.A.)—It is required to determine the singularity at the origin in the curve

$$(ax^2+bxy+cy^2+dx+ey)^2-(fx+gy)(hx+ky)(dx+ey)^2=0.$$

Solution by the PROPOSER.

Let Δ be the multiplicity of the singularity. If we suppose $fx+gy$ to coincide with $hx+ky$, we add one double point to the singularity of the curve. At the same time its equation can be decomposed into two equations of the degree 2. Hence

$$\Delta+1=4 \quad \text{or} \quad \Delta=3.$$

That is to say, we have three adjacent double points at the origin, two of which evidently lie on the tangent $dx+ey=0$.

Although the terms of the second order form a square, the origin is not a cusp. It will be noticed that the next higher terms contain the tangent as factor.

3295. (Proposed by R. W. GENESE.)—If Q be any point on a circle ABC, and the arcs be measured round the circle in the same direction;

then $\quad \sin BC \cos QA + \sin CA \cos QB + \sin AB \cos QC = 0,$

and $\quad \sin BC \sin QA + \sin CA \sin QB + \sin AB \sin QC = 0.$

Solution by the Rev. J. L. Kitchin, M.A.

1. Call the arcs AB, BC, CQ, AQ, β, γ, δ, α respectively; then

$$\sin BC \cos QA + \sin CA \cos QB + \sin AB \cos QC$$
$$= \sin\gamma \cos\alpha + \sin(\alpha+\delta)\cos(\alpha+\beta) + \sin\beta\cos(\alpha+\beta+\gamma)$$
$$= \tfrac{1}{2}\left\{\sin(\gamma-\alpha) + \sin(2\alpha+\beta+\delta) + \sin(\delta-\beta) + \sin(\alpha+2\beta+\gamma)\right\}$$
$$= \sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\left\{\cos\tfrac{1}{2}(3\alpha+\beta+\delta-\gamma) + \cos\tfrac{1}{2}(\alpha+3\beta+\gamma-\delta)\right\} = 0,$$

since we have $\alpha+\beta+\gamma+\delta = 2\pi$;

therefore $\sin BC \cos QA + \sin CA \cos QB + \sin AB \cos QC = 0.$

2. $\operatorname{Sin}\gamma \sin\alpha + \sin(\alpha+\delta)\sin(\alpha+\beta) + \sin\beta\sin(\alpha+\beta+\gamma)$

$$= \tfrac{1}{2}\left\{\cos(\alpha-\gamma) + \cos(\beta-\delta) - \cos(2\alpha+\beta+\delta) - \cos(\alpha+2\beta+\gamma)\right\}$$
$$= \tfrac{1}{2}\left\{\cos(\alpha-\gamma) - \cos(2\alpha+\beta+\delta) + \cos(\beta-\delta) - \cos(\alpha+2\beta+\gamma)\right\}$$
$$= \sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\left\{\sin(\qquad\qquad) + \sin(\qquad\qquad)\right\} = 0,$$

therefore $\sin BC \sin QA + \sin CA \sin OB + \sin AB \sin QC = 0.$

[Mr. Genese remarks that (2) is a new form of Euc. VI., Prop. D, and suggests (1); and that he verifies by solving the equations for

$$\sin BC : \sin CA : \sin AB,$$

which give $\sin(QB-QC) : \sin(QC-QA) : \sin(QA-QB)$, &c.]

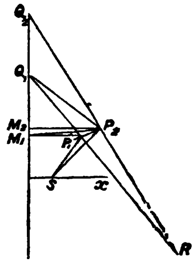
3394. (Proposed by C. Taylor, M.A.)—PQ is any diameter of a given ellipse, and R a point on a given confocal ellipse such that RP, RQ are equally inclined to the tangent at P. Show that RP + RQ is constant.

Solution by the Rev. R. Townsend, M.A., F.R.S.

If S be the point on the second ellipse diametrically opposite to R, then, since, from the equality of the angles in question, the four sides of the parallelogram PQRS touch a third given conic confocal with the other two, and since consequently (see Salmon's *Conic Sections*, 5th ed., Art. 399) the perimeter of the parallelogram, minus the circumference of the third ellipse, is constant, therefore, &c.

3309. (Proposed by C. W. Merrifield, F.R.S.)—Prove that the radius of curvature at any point of a parabola is double the portion of the normal intercepted between the curve and the directrix.

Solution by the PROPOSER.



Assuming that the focal distance is equal to the perpendicular on the directrix, and that the angle between these lines is bisected by the tangent, let RP_1Q_1 and RP_2Q_2 be consecutive normals. Then, since they are the external bisectors of SP_1M_1 and SP_2M_2, we shall always have

$$\angle P_1RP_2 = \tfrac{1}{2}\angle P_1SP_2 = \tfrac{1}{2}\angle P_1M_1P_2.$$

Further, since $Q_1P_2P_1$ and $Q_1M_1P_1$ are right angles, $M_1P_1P_2Q_1$ is an inscribed quadrilateral, and therefore $\angle P_1M_1P_2 = \angle P_1Q_1P_2$,

therefore $\quad \angle P_1RP_2 = \tfrac{1}{2}\angle P_1M_1P_2 = \tfrac{1}{2}P_1Q_1P_2.$

Hence, as P_1P_2 is small, we must have $P_1R = 2Q_1P_1$, or the radius of curvature is twice the length of the normal intercepted between the curve and the directrix.

[Another Solution is given on p. 37 of this volume of the *Reprint*.]

3251. (Proposed by the Rev. R. TOWNSEND, M.A., F.R.S.)—If a flexible chord of uniform thickness, in free equilibrium under the action of any forces which have a potential, intersect twice any surface of equilibrium of the forces, show that the two tensions are equal at the two intersections.

Solution by the PROPOSER.

Denoting, in the usual manner, by ϵ the constant area of the transverse section of the chord, and by X, Y, Z, and T the components of the acting forces and the tension at any point of it P; then, since

$$dT = -\epsilon\,(X\,dx + Y\,dy + Z\,dz) = -\epsilon\,dV,$$

therefore, for any two points P_1 and P_2 of it,

$$T_1 - T_2 = -\epsilon\,(V_1 - V_2),$$

and therefore, &c.

3217. (Proposed by W. C. OTTER, F.R.A.S.)—Find all the numbers of 7 figures each, and the sum of them all, that can be formed out of the number 1122222334567.

Solution by H. HOSKINS.

There are eleven different sets of numbers, of the following forms:—

1234567 .... (1),	2221345, &c...... (5),	2222113, &c...... (9),
1123456, &c...... (2),	2221134, &c...... (6),	2222213, &c...... (10),
1122345, &c...... (3),	2221133, &c...... (7),	2222211, &c...... (11).
1122334, &c...... (4),	2222134, &c...... (8),	

Hence the different numbers which can be formed from the sets are—

$$\underline{|7},\ 6\times3\times\frac{\underline{|7}}{\underline{|2}},\ \frac{5.4}{1.2}\times3\times\frac{\underline{|7}}{\{\underline{|2}\}^2},\ 4\times\frac{\underline{|7}}{\{\underline{|2}\}^3},\ \frac{6.5}{1.2}\times\frac{\underline{|7}}{\underline{|3}},$$

$$\frac{5.4}{1.2}\times2\times\frac{\underline{|7}}{\underline{|3}\,\underline{|2}},\ \frac{\underline{|7}}{\{\underline{|2}\}^2\,\underline{|3}},\ \frac{6.5.4}{1.2.3}\times\frac{\underline{|7}}{\underline{|4}},\ 5\times2\times\frac{\underline{|7}}{\underline{|4}\,\underline{|2}},$$

$$\frac{6.5}{1.2}\times\frac{\underline{|7}}{\underline{|5}},\ \text{and}\ 2\times\frac{\underline{|7}}{\underline{|5}\,\underline{|2}};\ \text{whence the sum of these}$$

$$=\underline{|7}\left(1+9+\frac{15}{2}+\frac{1}{2}+\frac{5}{2}+\frac{5}{3}+\frac{1}{24}+\frac{5}{6}+\frac{5}{24}+\frac{1}{8}+\frac{1}{120}\right)=23\tfrac{23}{60}\times\underline{|7}$$

$= 117852$, the different numbers which can be formed.

This result is very easily obtained from Art. 758 of TODHUNTER'S *Algebra*, and will be the coefficient of x^7 in the expression

$$(1+Px)^4\left(1+Px+\frac{P^2x^2}{1.2}\right)^2\left(1+Px+\frac{P^2x^2}{1.2}+\ldots\ldots+\frac{P^5x^5}{\underline{|5}}\right),$$

(where $P^7=\underline{|7}$); but it is necessary to obtain the different numbers in the first form, so as to arrive at their sum.

Now we must first find the number of times each digit can stand in any place in the numbers composing the different sets; and leaving out the factor $(1+10+10^2+\ldots+10^6)$ or $\dfrac{10^7-1}{10-1}$ to the last, it being common to all the sets, we find that each digit can occur as follows:—

In the first set $\underline{|6}$, and their sum $=(1+2+\ldots+7)\underline{|6}=28\underline{|6}$.

In the second set, 1, 2, 3 can occur $\left(6\times6\times\underline{|5}+5\times2\times\dfrac{\underline{|6}}{\underline{|2}}\right)=11\times\underline{|6}$;

4, 5, 6, 7 can occur $\quad 5\times3\times\dfrac{\underline{|6}}{\underline{|2}}=\dfrac{15}{2}\times\underline{|6}$; hence the sum is

$$\left\{11\times\underline{|6}(1+2+3)+\frac{15}{2}\times\underline{|6}\times(4+5+6+7)\right\}=231\times\underline{|6}.$$

Similarly, the sums in the third, fourth, ..., eleventh sets are

$$\left(6\times2\times\frac{5.4}{1.2}\times\frac{\underline{|5}}{\underline{|2}}+\frac{4.3}{1.2}\times\frac{\underline{|6}}{\underline{|2}\,\underline{|2}}\right)(1+2+3)+\left(\frac{4.3}{1.2}\times3\times\frac{\underline{|6}}{\{\underline{|2}\}^2}\right)\times(4+5+6+7)$$

$$=(69+99)\underline{|6}=168\underline{|6};$$

$$6\times4\times\frac{\underline{|5}}{\{\underline{|2}\}^2}(1+2+3)+\frac{\underline{|6}}{\{\underline{|2}\}^3}(4+5+6+7)=\left(6+\frac{11}{4}\right)\underline{|6}=\frac{35}{4}\times\underline{|6};$$

$$\left(\frac{6.5}{1.2}\right)^2\times\underline{|4}\times(2)+\frac{5.4}{1.2}\times\frac{\underline{|6}}{\underline{|3}}(1+3+4+5+6+7)$$

$$=\left(15+\frac{130}{3}\right)\underline{|6}=\frac{175}{3}\underline{|6};$$

$$\frac{6.5}{1.2}\times 2\times\frac{5.4}{1.2}\times\frac{\lfloor 4}{\lfloor 2}\times(2)+\left(6\times\frac{5.4}{1.2}\times\frac{\lfloor 5}{\lfloor 3}\right)+4\times\frac{\lfloor 6}{\lfloor 3\,\lfloor 2}(1+3)$$
$$+4\times 2\times\frac{\lfloor 6}{\lfloor 3\,\lfloor 2}(4+5+6+7)$$
$$=\left(10+8+\frac{44}{3}\right)\lfloor 6=\frac{98}{3}\lfloor 6;$$

$$\frac{6.5}{1.2}\times\frac{\lfloor 4}{\{\lfloor 2\}^2}\times(2)+6\times\frac{\lfloor 5}{\lfloor 3\,\lfloor 2}(1+3)=\left(\frac{1}{4}+\frac{1}{3}\right)\lfloor 6=\frac{7}{12}\lfloor 6;$$

$$\left(\frac{6.5.4}{1.2.3}\right)^2\times\lfloor 3\times(2)+\frac{5.4}{1.2}\times\frac{\lfloor 6}{\lfloor 4}(1+3+4+5+6+7)$$
$$=\left(\frac{20}{3}+\frac{65}{6}\right)\lfloor 6=\frac{35}{2}\lfloor 6;$$

$$\frac{6.5.4}{1.2.3}\times 5\times 2\times\frac{\lfloor 3}{\lfloor 2}\times(2)+\left\{6\times 5\times\frac{\lfloor 5}{\lfloor 4}+\frac{\lfloor 6}{\lfloor 4\,\lfloor 2}\right\}(1+3)$$
$$+2\times\frac{\lfloor 6}{\lfloor 4\,\lfloor 2}(4+5+6+7)$$
$$=\left(\frac{5}{3}+\frac{11}{12}+\frac{11}{12}\right)\lfloor 6=\frac{7}{2}\lfloor 6;$$

$$\left(\frac{6.5}{1.2}\right)^2\times\lfloor 2\times(2)+5\times\frac{\lfloor 6}{\lfloor 5}(1+3+4+5+6+7)$$
$$=\left(\frac{5}{4}+\frac{13}{12}\right)\lfloor 6=\frac{7}{3}\lfloor 6;$$

$$\frac{6.5}{1.2}\times 2\times(2)+6(1+3)=\frac{7}{60}\lfloor 6.$$

Hence the sum of all the numbers which can be formed

$$=\left(28+231+168+\frac{35}{4}+\frac{175}{3}+\frac{98}{3}+\frac{7}{12}+\frac{35}{2}+\frac{7}{2}+\frac{7}{3}+\frac{7}{60}\right)\times\frac{10^7-1}{10-1}\times\lfloor 6$$
$$=33047\times 12\times 1111111=440626622604.$$

The probable value of any number of 7 digits taken at random is

$$\frac{440626622604}{117852}=3738813{\cdot}2.$$

It is a singular thing that in the sum of the numbers not an *odd* digit enters, or the odds are 4095 to 1 against its so occurring.

3175. (Proposed by Dr. James Matteson.)—It is required to find four positive integral numbers, such that the sum of every two of them shall be a cube.

Solution by ASHER B. EVANS, M.A.

Let x_1, x_2, x_3, x_4 represent the four numbers, and put
$x_1+x_2=y_1^3,\ x_1+x_3=y_2^3,\ x_2+x_3=y_3^3,\ x_1+x_4=y_4^3,\ x_2+x_4=\beta^3,\ x_3+x_4=\phi^3.$
It is evident that all the conditions of the question will be satisfied if

$$\beta^3+y_2^3=\phi^3+y_1^3=y_3^3+y_4^3 \qquad \text{(1)},$$

the values of y_1, y_2, y_3, y_4 being so taken as to render x_1, x_2, x_3, x_4 positive integers.

Let us first satisfy the condition $\beta^3+y_2^3=\phi^3+y_1^3$ .... (2).
Put $\beta=(p+q)x$, $y_2=(p-q)x$, $\phi=(r+s)x$, $y_1=(r-s)x$; and (2) will become

$$p(p^2+3q^2)=r(r^2+3s^2) \qquad \text{(3)}.$$

The general values of p, q, r, s that will satisfy (3) are (see Euler's Algebra, Chap. XV.) $p=ft+3gu$, $q=gt-fu$, $r=ht+3ku$, $s=kt-hu$, where $u=f(f^2+3g^2)-h(h^2+3k^2)$, and $t=3k(h^2+3k^2)-3g(f^2+3g^2)$, the quantities f, g, h, k being taken at pleasure; therefore

$$\left.\begin{aligned}
\beta &= (p+q)x = [(3fk+3gk+fh-3gh)(h^3+3k^2)-(f^2+3g^2)^2]x\\
y_2 &= (p-q)x = [(3fk-3gk-fh-3gh)(h^2+3k^2)+(f^2+3g^2)^2]x\\
\phi &= (r+s)x = [(3fk-3gk-fh-3gh)(f^2+3g^2)+(h^2+3k^2)^2]x\\
y_1 &= (r-s)x = [(3fk+3gk+fh-3gh)(f^2+3g^2)-(h^2+3k^2)^2]x
\end{aligned}\right\}\ldots\text{(4)}.$$

Take $f=7$, $g=14$, $h=6$, $k=14$, and equations (4) will give

$$\beta=13x(1043),\ y_2=13x(2989),\ \phi=13x(1140),\ y_1=13x(2976)\ \ldots\text{(5)},$$

therefore $(1043)^3+(2989)^3=(1140)^3+(2976)^3=7^3.3^3.2^3.13.3613$ ...(6).
From (6) it is evident that (1) will be satisfied if we can find two cubes whose sum is 13·3613. Let $a=\frac{1}{2}(m+n)$ and $b=\frac{1}{2}(m-n)$ be the roots of those cubes; then $m(m^2+3n^2)=4.13.3613$ .... (7).
If integral values of m and n will satisfy (7), m must $=1, 2, 4, 13, 26$ or 52; take $m=13$, then $n=69$, therefore $a=41$, $b=-28$, and $13{\cdot}3613=41^3-28^3$. The root of one of these cubes being negative, we must seek further. Putting $m_1=41$ and $n_1=28$ in the identity

$$m_1^3-n_1^3=\left(\frac{m_1(m_1^3-2n_1^3)}{m_1^3+n_1^3}\right)^3+\left(\frac{n_1(2m_1^3-n_1^3)}{m_1^3+n_1^3}\right)^3,$$

we obtain $13{\cdot}3613=41^3-28^3=\left(\dfrac{341899}{30291}\right)^3+\left(\dfrac{1081640}{30291}\right)^3$ .... (8).

From (6) and (8) we have

$$(1043)^3+(2989)^3=(1140)^3+(2976)^3=\left(\frac{9573172}{10097}\right)^3+\left(\frac{30285920}{10097}\right)^3\ \ldots\text{(9)}.$$

By taking $13x=20194$, we find from (1), (4), and (9)

$$\beta=21062342,\quad y_2=60359866,\quad \phi=23021160,$$
$$y_1=60097344,\quad y_3=19146344,\quad y_4=60571840;$$

therefore

$$\begin{aligned}
x_1 &= \tfrac{1}{2}(y_1^3+y_2^3-y_3^3)=214972108693241589340948,\\
x_2 &= \tfrac{1}{2}(y_1^3+\beta^3-y_4^3)=2080913082956455142636,\\
x_3 &= \tfrac{1}{2}(y_3^3+\phi^3-\beta^3)=4937801347510680732948,\\
x_4 &= \tfrac{1}{2}(y_4^3+\beta^3-y_1^3)=7262810476410016163052.
\end{aligned}$$

3182. (Proposed by R. TUCKER, M.A.)—Circles are drawn touching the sides of a triangle and passing through the centre of the inscribed circle; if ρ_a, ρ'_a, &c., be the larger and smaller of these radii, prove (1), that the inscribed radius is a harmonic mean between them; and (2), that

$$\left[2\Sigma\left(\frac{1}{\rho}\right)-\Sigma\left(\frac{1}{\rho'}\right)\right]^2=\frac{18r}{\rho_a\rho_b\rho_c}.$$

Solution by STEPHEN WATSON; *the* PROPOSER; Rev. J. WOLSTENHOLME, M.A.; *and others.*

1. Let O be the inscribed centre; then

$$\text{AO} : r = \text{AO}+\rho_a : \rho_a,$$

therefore $\quad \dfrac{1}{\rho_a}=\dfrac{1}{r}-\dfrac{1}{\text{AO}}$; and similarly $\dfrac{1}{\rho'_a}=\dfrac{1}{r}+\dfrac{1}{\text{AO}}$;

therefore $$\frac{1}{\rho_a}+\frac{1}{\rho'_a}=\frac{2}{r}.$$

2. Since AO $\sin\frac{1}{2}\text{A}=r$, &c.,

therefore $$\left[2\Sigma\left(\frac{1}{\rho}\right)-\Sigma\left(\frac{1}{\rho'}\right)\right]^2=\frac{9}{r^2}\left(1-\sin\tfrac{1}{2}\text{A}-\sin\tfrac{1}{2}\text{B}-\sin\tfrac{1}{2}\text{C}\right)^2$$

$$=\frac{9}{r^2}\left(1-\sin\tfrac{1}{2}\text{A}\right)\left\{2-2\sin\tfrac{1}{2}\text{B}-2\sin\tfrac{1}{2}\text{C}-\sin\tfrac{1}{2}\text{A}+\cos\tfrac{1}{2}(\text{B}-\text{C})\right\},$$

which, since $\cos\frac{1}{2}(\text{B}-\text{C})-\sin\frac{1}{2}\text{A}=2\sin\frac{1}{2}\text{B}\sin\frac{1}{2}\text{C}$, becomes

$$=\frac{18}{r^2}\left(1-\sin\tfrac{1}{2}\text{A}\right)\left(1-\sin\tfrac{1}{2}\text{B}\right)\left(1-\sin\tfrac{1}{2}\text{C}\right)=\frac{18r}{\rho_a\rho_b\rho_c}.$$

3172. (Proposed by STEPHEN WATSON.)—Find the average areas of the circumscribed, inscribed, and escribed circles of the triangle FPf, where P is any point in the circumference of a given ellipse, and F, f the foci.

Solution by the PROPOSER.

Denote the point P by xy, and the several radii as usual, r_1, r_2, r_3 being the radii of those circles which touch Ff, fP, PF respectively. Let ϕ be the eccentric angle of P, and l the length of the quadrantal arc of the ellipse. Then

$$\text{R}=\frac{\text{PF}\,.\,\text{P}f}{2y}=\frac{a^2-e^2x^2}{2y},\quad r=\frac{cy}{\frac{1}{2}(\text{PF}+\text{P}f+\text{F}f)}=\frac{cy}{(a+c)},$$

$$r_1=\frac{cy}{a-c},\quad r_2=\frac{cy}{c+ex},\quad\text{and}\quad r_3=\frac{cy}{c-ex}\ \ldots\ldots\ldots\ldots\ (1).$$

From these we see that when P approaches one extremity of the major axis, R and r_2 approach *infinity*, and when the other extremity R and r_3

do the same; hence the average areas of the circles, radii R, r_2, r_3, are each *infinite*.

An element of the curve at P is $\dfrac{(a^4y^2+b^4x^2)^{\frac{1}{2}}}{a^2y}\,dx$; hence the average area of the circle whose radius is r, is

$$A=\frac{1}{l}\int_0^a \pi r^2\,\frac{(a^4y^2+b^4x^2)^{\frac{1}{2}}}{a^2y}\,dx=-\frac{\pi b^2c^2}{l\,(a+c)^2}\int_{\frac{1}{2}\pi}(b^2+c^2\sin^2\phi)^{\frac{1}{2}}\sin^2\phi\,d\phi$$

$$=-\frac{\pi b^3c^2}{l\,(a+c)^2}\int_{\frac{1}{2}\pi}^{0}d\phi\left\{1+\frac{1}{2}X-\frac{1}{2.4}X^2+\frac{1.3}{2.4.6}X^3-\&c.\right\}\sin^2\phi\;\ldots.(2),$$

where $X=\dfrac{c^2\sin^2\phi}{b^2}$. But $\displaystyle\int_{\frac{1}{2}\pi}^{0}\sin^{2n}\phi\,d\phi=-\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\,\tfrac{1}{2}\pi$;

hence (2) becomes

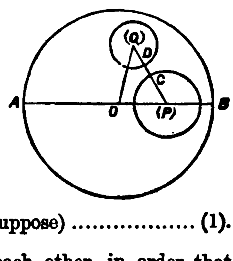
$$A=\frac{\pi^2b^3c^2}{2l\,(a+c)^2}\left\{\frac{1}{2}+\frac{1.3}{2^2.4}m^2-\frac{1^2.3.5}{2^2.4^2.6}m^4+\frac{1^2.3^2.5.7}{2^2.4^2.6^2.8}m^6-\&c.\right\},$$

where $m=\dfrac{c}{b}$. From this A can be found for any particular values of a and b.

From the above values of r, r_1, it is plain the average area of the circle whose radius is r_1 is $=\dfrac{a+c}{a-c}$ A.

3375 (Proposed by A. Martin.)—Two persons in a dark room draw each a circle at random on a circular slate; find the chance that the circumferences of the circles will intersect.

Solution by Stephen Watson.



Let O be the centre of the circular slate; P and Q those of the circles drawn at random; PC and QD their radii; and AB a diameter through P. The chance will be independent of the length of OB; put therefore OB=1, OP=x, PQ=y, PC=r, QD=ρ, and $\angle$ OPQ=ϕ. Then the condition that the circles, centres O, P, Q, touch each other, is $1-OQ=y-r$,

or $\qquad (1+r-y)^2=x^2+y^2-2xy\cos\phi$;

therefore $\qquad y=\dfrac{(1+r)^2-x^2}{2\,(1+r-x\cos\phi)}=m$ (suppose) .... (1).

Hence when the circles (P) and (Q) lie outside each other, in order that they may *not* intersect when $y>m$ but $<X+x\cos\phi$, ρ must lie between 0 and $1-(x^2+y^2-2xy\cos\phi)^{\frac{1}{2}}$; and when y is $>r$ but $<m$, ρ must lie be-

tween 0 and $y-r$, where $X^2 = 1-x^2 \sin^2\phi$. After integrating for y, we must put $\pi-\phi$ for ϕ, and add the results, to include the case of Q below AB. The other limits are as follows:—ϕ from 0 to $\frac{1}{2}\pi$, and doubled; r from 0 to $1-x$, and x from 0 to 1. Hence the number of ways the two circles (P) and (Q) can be drawn entirely outside each other is

$$\int_0^1 2\pi x\,dx \int_0^{1-x} dr \int_0^{\frac{1}{2}\pi} 2d\phi \left\{ \int_r^m y\,dy\,(y-r) + \int_m^{X+x\cos\phi} y\,dy \right.$$
$$\left. \times \left[1-(x^2+y^2-2xy\cos\phi)^{\frac{1}{2}}\right] \right\} \quad\text{.........(2).}$$

Integrated for y, this becomes

$$r\left(1+r+\tfrac{1}{2}r^2\right)-\tfrac{1}{2}x^2\cos^2\phi\,(1+r)+\tfrac{1}{8}\cdot\frac{\left\{(1+r)^2-x^2\right\}^2}{1+r-x\cos\phi}$$
$$+\tfrac{1}{2}\left(X^2+x\cos\phi\,X+x^2\cos^2\phi\right)-\tfrac{1}{2}\left(1+r+\tfrac{1}{2}x\cos\phi\right)\left\{(1+r)^2-x^2\right\}$$
$$-\tfrac{1}{2}x^3\sin^2\phi\cos\phi\log\frac{1+X}{1+r-x\cos\phi} \quad\text{.....................(3).}$$

Putting $\pi-\phi$ for ϕ, and adding, (2) becomes

$$4\pi\int_0^1 x\,dx\int_0^{1-x} dr\int_0^{\frac{1}{2}\pi} d\phi\left\{2r\left(1+r+\tfrac{1}{2}r^2\right)-(1+r)\,x^2\cos^2\phi\right.$$
$$+\tfrac{1}{4}(1+r)\frac{\left[(1+r)^2-x^2\right]^2}{(1+r)^2-x^2\cos^2\phi}+\left(X^2+x^2\cos^2\phi\right)-(1+r)\left[(1+r)^2-x^2\right]$$
$$\left.+\tfrac{1}{2}x^3\sin^2\phi\cos\phi\log\frac{1+r-x\cos\phi}{1+r+x\cos\phi}\right\}$$
$$=4\pi^2\int_0 x\,dx\int_0^{1-x} dr\left\{\tfrac{1}{6}\left(1+r^3\right)-\tfrac{1}{24}\left[(1+r)^2-x^2\right]^{\frac{3}{2}}\right\},$$

which, by changing the order of integration, becomes

$$=\pi^2\int_0^1 dr\int_0^{1-r} x\,dx\left\{\tfrac{2}{3}\left(1+r^3\right)-\tfrac{1}{6}\left[(1+r)^2-x^2\right]^{\frac{3}{2}}\right\}$$
$$=\pi^2\int_0^1 dr\left\{\tfrac{1}{3}\left(1+r^3\right)(1-r)^2-\tfrac{1}{30}(1+r)^5+\tfrac{16}{15}r^{\frac{5}{2}}\right\}=\frac{\pi^2}{14}.$$

Again, the number of ways the circle (Q) can lie within (P) is

$$\int_0^1 2\pi x\,dx\int_0^{1-x} dr\int_0^{\frac{1}{2}\pi} 2d\phi\int_0^r 2y\,dy\,(r-y)=\frac{\pi^2}{180};$$

and doubling this, because either circle may lie within the other, the total number of ways the circles (P) and (Q) can be drawn not to intersect is

$$\pi^2\left(\tfrac{1}{14}+\tfrac{1}{90}\right)=\tfrac{26}{315}\pi^2.$$

But the circles (P) and (Q) can be drawn in $\frac{1}{6}\pi^2$ ways; therefore the chance of non-intersection is $\frac{26}{35}$, and of intersection $\frac{9}{35}$.

2752. (Proposed by R. TUCKER, M.A.)—Through a point O, within a triangle ABC, parallels DE, FG, HI are drawn to BC, CA, AB respectively; find the locus of O when one of the three triangles AGH, BDI, CFE, external to the hexagon GDIFEH, is (1) equal to the sum of the other two, or (2, 3, 4) an arithmetic, geometric, or harmonic mean between them. (See *Reprint*, Vol. IX., pp. 25, 41.)

Solution by the PROPOSER.

With the notation of my solution (*Reprint*, Vol. IX., p. 26), we have

$$2\Delta DBI = xy \sin B \text{ ....(i)},$$
$$2\Delta AGH = b(1-n)(nc-y)\sin A \text{ ....(ii)},$$
$$2\Delta CEF = \frac{aby}{c^2}(nc-y)\sin C \text{ ....(iii)}.$$

1. Then, if in all cases the single triangle be DBI, the locus is

$$bxy = ab(1-n)(nc-y) + \frac{aby}{c}(nc-y), \quad \text{or} \quad a^2y^2 + 3acxy + c^2x^2 = ac^2x + a^2cy,$$

an hyperbola circumscribing the triangle, with centre at the point $(\frac{1}{5}a, \frac{1}{5}c)$.

2. If m (i) = (ii) + (iii), we have

$$a^2y^2 + (m+2)\,acxy + c^2x^2 = ac^2x + a^2cy;$$

hence, for all positive values of m, we have the portion of a hyperbola circumscribing the triangle contained within the triangle.

3. Here the locus is $b^2x^2y^2 = \frac{b^2}{c}(1-n)\,a^2y\,(nc-y)^2$,

or $$a^2y^2 + acxy + c^2x^2 - 2a^2cy - 2ac^2x + a^2c^2 = 0;$$

a portion of an ellipse touching the sides at A, C, with centre $(\frac{2}{3}a, \frac{2}{3}c)$.

4. In this case, making the necessary substitutions, we have

$$x[c(1-n)+y] = 2a(1-n)(nc-y), \quad \text{that is,} \quad cx + ay = \tfrac{2}{3}ac,$$

or a straight line through the centre of gravity of the triangle parallel to AC.

Lastly, if λ (i) + μ (ii) + ν (iii) is constant, the locus of O will be found

from $$\lambda cxy + \mu cx(nc-y) + \nu ay(nc-y) = \text{a constant} = k^3,$$

or $$\mu c^2x^2 + (\mu+\nu-\lambda)\,acxy + \nu a^2y^2 = \mu c^2ax + \nu a^2cy.$$

[In the case of (1), $\mu = \nu = -\lambda = 1$.]

3241. (Proposed by R. TUCKER, M.A.)—If, in a spherical triangle, $A = \frac{2}{5}\pi$, $B = \frac{3}{5}\pi$, $C = \frac{1}{2}\pi$, then

$$\tan r_1 \tan r_2 = \tan^2 r_3 = \tan^2 r, \quad \tan R_1 \tan R_2 = 2, \quad \tan^2 r(\sec^2 R + 1) = 2.$$

Solution by the PROPOSER.

The first property is readily seen to hold for all right-angled spherical triangles which have their other angles supplementary, for

$$\tan r_1 = \tan \tfrac{1}{2}A \sin s, \quad \tan r_2 = \cot \tfrac{1}{2}A \sin s;$$

hence $$\tan r_1 \tan r_2 = \sin^2 s = \tan^2 r_3 \text{ ....(1)},$$

and it is also readily shown that $a+b=\pi$, whence $r=r_3$.

Again, $\tan R_1 = \dfrac{\cos(S-A)}{N} = \dfrac{\cos 63°}{N} = \dfrac{\sin 27°}{N}$, $\tan R_2 = \dfrac{\cos 27°}{N}$,

and $N = \left\{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)\right\}^{\frac{1}{2}} = (\frac{1}{2}\sin 27°\cos 27°)^{\frac{1}{2}}$;

therefore $$\tan R_1 \tan R_2 = 2 \qquad\qquad (2).$$

Also, $$\tan r = \sin s = \cos \tfrac{1}{2}c, \quad \tan R = \sqrt{2}\tan\tfrac{1}{2}c;$$

therefore $$\tan^2 r\tan^2 R = 2\sin^2\tfrac{1}{2}c = 2-2\tan^2 r,$$

or $$\tan^2 r(\sec^2 R+1) = 2 \qquad\qquad (3).$$

3173. (Proposed by J. J. WALKER, M.A.)—In any spherical triangle ABC, let D be the middle point of the arc BC, and E another point in BC such that the angle BAE is equal to the angle DAC; also let AF be the arc through A perpendicular to BC; prove that

$$(1)\ \frac{\cos AEB}{\cos ADB} = -\cos(B+C);\quad (2)\ \tan\tfrac{1}{2}DAE = \frac{\tan\frac{1}{2}(b-c)}{\tan\frac{1}{2}(b+b)}\tan\frac{A}{2};$$

$$(3)\ \tan EAF = \frac{\sin^2\frac{1}{2}c-\sin^2\frac{1}{2}b}{\sin^2\frac{1}{2}b\cos^2\frac{1}{2}c+\sin^2\frac{1}{2}c\cos^2\frac{1}{2}b}\cot(B+C).$$

Solution by R. TUCKER, M.A.

1. Let $\angle DAC = \theta = \angle BAE$, then

$$\cos ADB = \cos C\cos\theta - \sin C\sin\theta\cos b,$$
$$\cos AEB = -\cos B\cos\theta + \sin B\sin\theta\cos c,$$

and $$\sin b\sin\theta = \sin ADB\sin\tfrac{1}{2}a = \sin c\sin(A-\theta),$$

whence $$\cot\theta = \frac{\sin b+\sin c\cos A}{\sin c\sin A};$$

$$\therefore\ \frac{\cos AEB}{\cos ADB} = \frac{\sin B\cos c - \cos B\cot\theta}{\cos C\cot\theta - \sin C\cos b}$$
$$= \frac{\sin B\cos c\sin C\sin A - \sin B\cos B - \cos B\sin C\cos A}{\sin B\cos C + \sin C\cos A\cos C - \sin^2 C\cos b\sin A}$$
$$= \frac{\sin 2C - \sin 2B}{2(\sin B\cos C - \sin C\cos B)} = -\cos(B+C).$$

2. Next $$\tan\tfrac{1}{2}DAE = \tan(\tfrac{1}{2}A-\theta) = \frac{\tan\frac{1}{2}A - \dfrac{\sin c\sin A}{\sin b+\sin c\cos A}}{1+\tan\frac{1}{2}A\dfrac{\sin c\sin A}{\sin b+\sin c\cos A}}$$
$$= \frac{\sin b-\sin c}{\sin b+\sin c}\tan\tfrac{1}{2}A = \frac{\tan\frac{1}{2}(b-c)}{\tan\frac{1}{2}(b+c)}\tan\tfrac{1}{2}A.$$

3. Again, let $AF=\delta$, and $\angle FAE=\phi$; then
$$\cos B = \cos\delta\sin(\theta-\phi),\quad \cos C = \cos\delta\sin(A-\theta+\phi);$$

hence $\cot\frac{1}{2}(B+C)\cot\frac{1}{2}(B-C) = \tan\frac{1}{2}A\cot(\frac{1}{2}A-\theta+\phi)$,

or $\tan\frac{1}{2}A\dfrac{\cos\frac{1}{2}(b+c)\sin\frac{1}{2}(b+c)}{\cos\frac{1}{2}(b-c)\sin\frac{1}{2}(b-c)} = \dfrac{\tan\frac{1}{2}(b+c)-\tan\frac{1}{2}(b-c)\tan\frac{1}{2}A\tan\phi}{\tan\frac{1}{2}(b-c)\tan\frac{1}{2}A+\tan\frac{1}{2}(b+c)\tan\phi}$,

$$\tan\phi\cos\tfrac{1}{2}(b+c)\left[\frac{\sin^2\frac{1}{2}(b+c)+\sin^2\frac{1}{2}(b-c)}{\sin\frac{1}{2}(b-c)\cos\frac{1}{2}(b-c)\sin\frac{1}{2}(b+c)}\right]$$
$$= \frac{\cos\frac{1}{2}(b+c)}{\cos\frac{1}{2}(b-c)}\left[\tan\tfrac{1}{2}(B+C)-\cot\tfrac{1}{2}(B+C)\right]$$

(TODHUNTER'S *Spherical Trigonometry*, Art. 52);

that is, $\tan\phi(\sin^2\frac{1}{2}b\cos^2\frac{1}{2}c+\cos^2\frac{1}{2}b\sin^2\frac{1}{2}c)$
$$= \tfrac{1}{2}(\cos c-\cos b)\cot(B+C) = (\sin^2\tfrac{1}{2}b-\sin^2\tfrac{1}{2}c)\cot(B+C).$$

3011. (Proposed by R. TUCKER, M.A.)—Two points are taken on a parabola such that the normals intersect on the evolute; then (1) the chord through them envelopes a parabola passing through the vertex; (2) the envelope of the circle through the points and the vertex is a quartic curve passing through the vertex.

Solution by the PROPOSER.

Let (α, β) be the common point of the normals, y the ordinate of one of the points, and $y^2 = 4ax$ the parabola; then we can readily get for the determination of y the equation
$$y^3-4a(\alpha-2a)y-8a^2\beta = 0.$$
Now two of the values coincide; hence
$$y_1^2+2y_1y_3 = -4a(\alpha-2a),\quad y_1^2y_3 = 8a^2\beta,\quad 2y_1+y_3 = 0.$$
From these we obtain $y_1^2 = \frac{4}{3}a(\alpha-2a)$, $\quad x_1 = \frac{1}{3}(\alpha-2a)$,
$$y_3^2 = \tfrac{16}{3}a(\alpha-2a),\quad x_3 = \tfrac{4}{3}(\alpha-2a);$$
hence the equation to the chord is
$$y-\left(\frac{4a(\alpha-2a)}{3}\right)^{\frac{1}{2}} = -\left(\frac{12a}{\alpha-2a}\right)^{\frac{1}{2}}\left(x-\frac{\alpha-2a}{3}\right),$$
or
$$y\left(\frac{\alpha-2a}{12a}\right)^{\frac{1}{2}}+x = \tfrac{2}{3}(\alpha-2a).$$
Differentiating, we obtain $\dfrac{y}{(12a)^{\frac{1}{2}}}\cdot\dfrac{1}{(\alpha-2a)^{\frac{1}{2}}} = \dfrac{4}{3}$,
and for the envelope required $y^2+32ax = 0$, a parabola having the same vertex and axis, concavity in opposite direction, and parameter equal 8 times given parameter.

The equation to the circle (as in Question 2902 (*Reprint*, Vol. XI., p. 104) is $\quad x^2+y^2-(\alpha+2a)x-\frac{1}{2}\beta y = 0$ .... (1),
and we have the condition $\quad 27a\beta^2 = 4(\alpha-2a)^3$ .... (2).

From (1), $\frac{d\beta}{d} = -\frac{2x}{y}$; therefore, from (2), $-9a\beta x = y(\alpha - 2a)^2$,

and $$\beta = -16a\frac{x^3}{y^3}, \quad \alpha - 2a = \pm 12a\frac{x^2}{y^2}.$$

Putting these values in (1), we have for the envelope

$$(x^2+y^2)\,y^2 \mp 12ax^3 - 2axy^2 + 8ax^3 = 0.$$

3401. (Proposed by Professor SYLVESTER.)—Let Q be the quadratic invariant of $(a, b, c \ldots\ldots h, k)(x, y)^{2m}$, and Q′ of $(b, c \ldots\ldots h, k, l)(x, y)^{2m}$, and let E represent $a\delta_b + 2b\delta + \ldots\ldots + (2m+1)\,k\delta_l$: show that

$$2Q = E\,.\,E\,.\,Q'.$$

Solution by J. J. WALKER, M.A.

Writing $a_0, a_1, a_2, \ldots a_{2m-1}, a_{2m}, a_{2m+1}$ for $a, b, c, \ldots h, k, l$ respectively,

$$Q = a_0 a_{2m} - 2ma_1 a_{2m-1} + \frac{2m(2m-1)}{1\,.\,2}\,a\;a_{2m-1}\ldots\ldots$$

$$\ldots\ldots + (-1)^{m-1}\frac{2m(2m-1)\ldots m+2}{1\,.\,2\ldots m-1}\,a_{m-1}a_{m+1}$$

$$+ \tfrac{1}{2}(-1)^m\frac{2m(2m-1)\ldots m+1}{1\,.\,2\ldots m}\,a_m^2,$$

and Q′ is what Q becomes when the suffixes in every term are increased by 1. Hence

$$E\,.\,Q' = a_0 a_{2m+1} + (2m+1)\,a_1 a_{2m} - \frac{2m}{1}\,(2a_1 a_{2m} + 2ma\;a_{2m-1})$$

$$+ \frac{2m(2m-1)}{1\,.\,2}\left\{3a_2 a_{2m-1} + (2m-1)\,a_3 a\;{}_{-2}\right\}\ldots\ldots$$

$$\ldots\ldots + (-1)^{m-1}\frac{2m(2m-1)\ldots m+2}{1\,.\,2\ldots m-1}\left\{ma_{m-1}a_{m+2} + (m+2)\,a_m a_{m+1}\right\}$$

$$+ (-1)^m\frac{2m(2m-1)\ldots m+1}{1\,.\,2\ldots m}\,a_m a_{m+1}$$

$$= a_0 a_{2m+1} - (2m-1)\,a_1 a_{2m} + \frac{2m(2m-3)}{1\,.\,2}\,a_2 a_{2m-1}$$

$$-\frac{2m(2m-1)(2m-5)}{1\,.\,2\,.\,3}\,a_3 a_{2m-2}\ldots\ldots$$

$$\ldots\ldots + (-1)^{m-1}\frac{2m(2m-1)\ldots(m+3)\,3}{1\,.\,2\ldots(m-2)(m-1)}\,a_{m-1}a_{m+2}$$

$$+ (-1)^m\frac{2m(2m-1)\ldots(m+2)\,1}{1\,.\,2\ldots(m-1)\,m}\,a_m a_{m+1}.$$

After the second, the rth term of this series is

$$(-1)^{r-1}\frac{2m(2m-1)...(2m-r+3)(2m-2r+3)}{1.2\dots(r-2)(r-1)}a_{r-1}a_{2m-r+2},$$

which gives rise in E . E . Q′ to the terms

$$(-1)^{r-1}\frac{2m\dots 2m-2r+3}{1\dots m-1}\left\{(r-1)\,a_{r-2}\,a_{2m-r+2}+(2m-r+2)\,a_{r-1}\,a_{2m-r+1}\right\},$$

the consecutive being

$$(-1)^{r}\,\frac{2m\,(2m-1)\dots(2m-2r+1)}{1.2\dots(r-1)\,r}\left\{ra_{r-1}a_{2m-r+1}+(2m-r+1)\,a_r\,a_{2m-r}\right\}.$$

Hence in E . E . Q′ the coefficient of $a_{r-1}\,a_{2m-r+1}$ is

$$(-1)^{r-1}\frac{2m\dots 2m-r+2}{1\dots r-1}\left\{2m-2r+3-(2m-2r+1)\right\},$$

that is, twice the coefficient of $a_{r-1}\,a_{2m-r+1}$ in Q. The first and second terms in E . Q′ give rise in E . E . Q′ to

$$(2m+1)\,a_0\,a_{2m}-(2m-1)\,a_0\,a_{2m}-2m\,(2m-1)\,a_1\,a_{2m-1}, \text{ or } 2a_0\,a_{2m}-\dots\dots,$$

the latter of which combines with the first of the pair of terms in E.E.Q′ to which the third term in E.Q′ gives rise. Finally, the last term in E.Q′ gives rise in E.E.Q′ to the term

$$(-1)^m\,\frac{2m\,(2m-1)\dots m+1}{1.2\dots m}\,a_m^2,$$

besides one which combines with one of the terms arising from operating with E on the last term but one of E.Q′.

3266. (Proposed by J. F. MOULTON, M.A.)—A cone has its vertex at the origin, and intersects the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ along the curve of contact with it of the tangent cone whose vertex is $f, g, 0$. Show that, if the first cone be a right one, the point lies on a conic which becomes a rectangular hyperbola if $c^2(a^2+b^2)=a^4+b^4$.

Solution by the Rev. G. H. HOPKINS, M.A.

It is easily seen that the cone with origin for vertex, and passing through the curve of contact $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1,\;\; \dfrac{xf}{a^2}+\dfrac{yg}{b^2}=1$, is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=\left(\frac{xf}{a^2}+\frac{yg}{b^2}\right)^2 \dots\dots\dots\dots\dots\dots\dots\dots(1).$$

Now assume $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$ to be the equation to the axis of this; then $\dfrac{x^2+y^2+z^2}{(lx+my+nz)^2}$ is constant; (x, y, z) being a point upon the cone, and the

cone being a right one; therefore

$$x^2+y^2+z^2 = k(lx+my+nz)^2 \qquad\qquad (2).$$

The equations (1) and (2) must be identical; therefore

$$\frac{1}{a^2}-\frac{f^2}{a^4} = \lambda(1-kl^2),\quad \frac{fg}{a^2b^2} = \lambda klm,\quad \frac{1}{b^2}-\frac{g^2}{b^4} = \lambda(1-km^2),\quad 0 = kln,$$

$$\frac{1}{c^2} = \lambda(1-kn^2),\quad 0 = kmn;\ \text{from which } n=0,\ \lambda = \frac{1}{c^2};\ \text{and}$$

$$\left\{\frac{c^2}{a^2}\left(1-\frac{f^2}{a^2}\right)-1\right\}\left\{\frac{c^2}{b^2}\left(1-\frac{g^2}{b^2}\right)-1\right\} = \frac{c^4f^2g^2}{a^4b^4},$$

or
$$\frac{c^2}{a^4}\left(1-\frac{c^2}{b^2}\right)f^2+\frac{c^2}{b^4}\left(1-\frac{c^2}{a^2}\right)g^2+\text{a constant}=0,$$

a conic section which becomes a rectangular hyperbola when

$$\frac{b^2-c^2}{a^2} = -\frac{a^2-c^2}{b^2},\quad\text{or}\quad a^4+b^4 = c^2(a^2+b^2).$$

3406. (Proposed by the EDITOR.)—Find x, y, z from the equations

$$x^2(y-1) = x+z,\quad x(y-x^2) = 1-z,\quad yz(y+1) = 3(1+2z).$$

I. *Solution by* MISS GREAVES.

From the three equations we get

$$z = x^2y-x^2-x = 1-xy+x^3 = \frac{3}{y^2+y-6}.$$

Equating the first two values of z, we get $y = x+\frac{1}{x}$. Then equating the second and third values of z, substituting for y, and reducing, we get

$$x^2(x^5-5x^3+5x^2-4) = 0.$$

The values of x which satisfy this equation are

$$0,\quad 1{\cdot}69411,\quad -{\cdot}70022,\quad -2{\cdot}58888,$$

and two imaginary roots which may be found approximately from the quadratic
$$x^2-1{\cdot}595x+1{\cdot}30246 = 0.$$

The values of x being thus known, y and z may be found from the expressions
$$y = x+\frac{1}{x},\quad z = x^2(x-1).$$

II. *Solution by* SAMUEL BILLS.

(1) + (2) gives
$$x^2y+xy-x^3-x^2 = 1+x,$$
or $xy(x+1)-x^2(x+1)-(x+1) = 0$, or $(x+1)(xy-x^2-1) = 0 \ldots (4)$.

Taking first $x+1=0$, we have $x=-1$; and from either (1) or (2) we

find $y = z$, and by substitution in (3) we obtain $y^3 + y^2 - 6y - 3 = 0$ ......(5). The solution of (5) gives one solution of the question.

Again, taking the other factor of (4), we have $xy - x^2 - 1 = 0$, whence $y = x + \frac{1}{x}$, and from (1), $z = (y-1)x^2 - x = x^3 - x^2$. Substituting these results in (3), we get the equation $x^5 - 5x^3 + 5x^2 - 4 = 0$ ....(6).

x being obtained from (6), we shall have $y = x + \frac{1}{x}$, and $z = x^2(x-1)$. It appears the question admits of six solutions.

3411. (Proposed by Disco.)—Show that $\left(\frac{n}{n+1}\right)^n$ is the probability that the product of n positive random quantities is greater than the nth power of one positive random quantity. What is the limiting value of this probability when n becomes infinite?

I. *Solution by* Hugh M'Coll.

Let $x_1, x_2, x_3, \ldots, x_n$ be the n positive random quantities; and let y be the one positive random quantity. The probability that $y^n < x_1 x_2 x_3 \ldots x_n$ is equivalent to the probability that $y < (x_1 x_2 x_3 \ldots x_n)^{\frac{1}{n}}$. First, suppose all the variables to be restricted between 0 and some finite limit a; and let Q_0 denote the probability of the event above stated when y alone is taken at random; Q_1 the probability when y and x_1 alone are taken at random; Q_2 the probability when y, x_1, and x_2 are alone taken at random; and so on. Then we shall have

$$Q_0 = \frac{1}{a}(x_1 x_2 x_3 \ldots x_n)^{\frac{1}{n}} \qquad = \left(\frac{x_1}{a} \cdot \frac{x_2}{a} \cdot \frac{x_3}{a} \cdots \frac{x_n}{a}\right)^{\frac{1}{n}};$$

$$Q_1 = \frac{1}{a}\int_0^a Q_0\, px_1 = \frac{1}{a}\int_0^a Q_0\, dx_1 = \left(\frac{n}{n+1}\right)\left(\frac{x_2}{a} \cdot \frac{x_3}{a} \cdots \frac{x_n}{a}\right)^{\frac{1}{n}};$$

$$Q_2 = \frac{1}{a}\int_0^a Q_1\, px_2 = \frac{1}{a}\int_0^a Q_1\, dx_2 = \left(\frac{n}{n+1}\right)^2\left(\frac{x_3}{a} \cdot \frac{x_4}{a} \cdots \frac{x_n}{a}\right)^{\frac{1}{n}};$$

$$Q_r = \frac{1}{a}\int_0^a Q_{r-1}\, px_r = \frac{1}{a}\int_0^a Q_{r-1}\, dx_r = \left(\frac{n}{n+1}\right)^r\left(\frac{x_{r+1}}{a} \cdot \frac{x_{r+2}}{a} \cdots \frac{x_n}{a}\right)^{\frac{1}{n}}.$$

$$\text{Hence } Q_n = \frac{1}{a}\int_0^a Q_{n-1}\, px_n = \frac{1}{a}\int_0^a \left(\frac{n}{n+1}\right)^{n-1}\left(\frac{x_n}{a}\right)^{\frac{1}{n}} px_n = \left(\frac{n}{n+1}\right)^n.$$

This result, being independent of the value of a, is true when a is infinite, so that the required proof has been given.

Again, $\left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n}$, and the limiting value of this when n becomes infinite is $\frac{1}{e}$, in which e, as usual, denotes the base of the Napierian logarithms.

[For the meaning of the symbol px_r, see Mr. M'Coll's article on *Probability Notation* in the *Educational Times* for January.]

II. *Solution by* J. Hopkinson, D.Sc., B.A.

Let $x_1, x_2, \ldots, x_n$ be n random quantities, and y the other random quantity, all values being assumed equally probable.

Let us first take x_1 between 0 and a_1, &c., and y between 0 and b, and afterwards make these quantities infinite.

Denote the continued product $x_1 \ldots x_n$ by $P(x)$; then the probability that $P(x) > y^n$ is

$$\frac{\int_0^{a_1}\int_0^{a_2}\ldots\ldots\int_0^{a_n}\int_0^{\{P(x)\}^{\frac{1}{n}}} dx_1 \ldots\ldots dx_n\, dy}{P(a)\,.\,b}$$

$$= \frac{\int_0^{a_1}\ldots\ldots\int_0^{a_n}\{P(x)\}^{\frac{1}{n}}\, dx_1 \ldots\ldots dx_n}{P(a)\,.\,b} = \left(\frac{n}{n+1}\right)^n \frac{\{P(a)\}^{\frac{1}{n}}}{b}.$$

Now the value of this, when $a_1 \ldots a_n$ and b become infinite, depends altogether upon the relation between the a's and b as they become infinite. If we suppose b to become infinite first, or to be of a higher order than the a's, the result is zero; and by assuming a proper relation between them, any result can be obtained. But we may perhaps fairly assume that the a's and b are equal; then the probability $= \left(\frac{n}{n+1}\right)^n$.

When n is infinite, the limiting value of this is clearly ϵ^{-1}.

3368. (Proposed by R. Tucker, M.A.)—A triangle is formed by joining the escribed centres of a given triangle, the same is done for this new triangle, and so on for ever; prove that the ultimate triangle is an equilateral one.

I. *Solution by the* Proposer.

Let $A_1, A_2, A_3, \ldots$ be successive corresponding angles of the triangles; then we have $A_1 = \frac{1}{2}\pi - \frac{1}{2}A$, $A_2 = \frac{1}{4}\pi + \frac{1}{4}A$, $A_3 = \frac{3}{8}\pi - \frac{1}{8}A$, &c.

The general value $A_n = \frac{2^{n+1}-(-1)^{n-1}}{3}\,\frac{\pi}{2^{n+1}} + (-1)^n \frac{A}{2^n}$.

Hence, in the case of the ultimate triangle, when n is made indefinitely great, we have each angle $= \frac{1}{3}\pi$, which proves the property.

[Mr. Tucker remarks that the same series arises in the case given in Todhunter's *Trigonometry*, Chap. XVI., Ex. 16, where the points of contact of the successive inscribed circles are joined to four new triangles, as can readily be inferred by drawing the figure, when we shall have two series of triangles always similar, so that the limit for both is the same, viz. the equilateral.]

II. *Solution by the* Rev. R. Townsend, M.A., F.R.S.; S. Watson; *and others.*

If A, B, C be the three angles of the original triangle, A′, B′, C′ the corresponding three of the first derived triangle, A″, B″, C″ the corresponding three of the second derived triangle, and so on; then, since evidently $A' = \frac{1}{2}(B+C)$, $B' = \frac{1}{2}(C+A)$, $C' = \frac{1}{2}(A+B)$,
and since consequently

$$(B'-C') = \tfrac{1}{2}(C-B), \quad (C'-A') = \tfrac{1}{2}(A-C), \quad (A'-B') = \tfrac{1}{2}(B-A),$$

and similarly

$$(B''-C'') = \tfrac{1}{2}(C'-B'), \quad (C''-A'') = \tfrac{1}{2}(A'-C'), \quad (A''-B'') = \tfrac{1}{2}(B'-A'),$$

and so on; therefore, &c.

3292. (Proposed by H. McColl.)—From the centre C of a square straight lines CP_1, CP_2, CP_3, &c., are drawn at equal angular intervals to meet the perimeter in the points P_1, P_2, P_3, &c. Show that (the number of these lines being increased without limit) the average area of the circle of which the variable CP is the radius is equal to the area of the given square. Show that this is also true if we substitute equilateral and equiangular polygon for the word square.

Solution by Artemas Martin.

Put $2a$ = a side of the square, and let $\angle PCD = \phi$; then CD (the perpendicular from C on one side) $= a$, $CP = a \sec\phi$, the area of the circle whose radius is $CP = \pi a^2 \sec^2\phi$, and its average area is

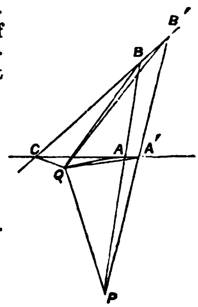
$$\frac{4}{\pi}\int_0^{\frac{1}{4}\pi} \pi a^2 \sec^2\phi\, d\phi = \Big[4a^2 \tan\phi\Big]_0^{\frac{1}{4}\pi} = 4a^2 = \text{area of the given square.}$$

For a regular polygon of n sides, putting $CD = a$, the average area of the circle (CP) is

$$\frac{n}{\pi}\int_0^{\frac{\pi}{n}} \pi a^2 \sec^2\phi\, d\phi = \Big[n a^2 \tan\phi\Big]_0^{\frac{\pi}{n}} = na^2 \tan\frac{\pi}{n} = \text{area of the polygon.}$$

3338. (Proposed by W. H. H. Hudson, M.A.)—A straight line moving according to some law intersects two fixed straight lines CA, CB in A, B, and itself ultimately in P; CQ is the chord of ultimate intersection of circles described round CAB. Prove that the anharmonic ratio

$$C\,(QBPA) = \left(\frac{\sin A}{\sin B}\right)^2.$$



Solution by the Rev. F. D. Thomson, M.A.

It is easily seen that BQB′ = AQA′ = APA′; therefore circles may be described round B′BQP, QAA′P.

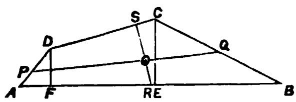
Again, $\quad C\,(QBPA) = Q\,(CBPA)$

$$= \frac{\sin CQB \sin PQA}{\sin CQA \sin BQP} = \frac{\sin A \sin PA'A}{\sin B \sin BB'P} = \frac{\sin^2 A}{\sin^2 B}$$

ultimately when A′B′ moves up to AB.

3288. (Proposed by S. Bills.)—Given a quadrilateral whose base is 8, and the perpendiculars thereon from its opposite angles 1 and 2, dividing the base into segments 1, 3, 4; it is required to divide the quadrilateral into four equal parts by two straight lines at right angles to each other.

Solution by the Proposer.



Let ABCD be the given quadrilateral. Draw CE, DF perpendicular to AB. By the question, AF = DF = 1, FE = 3, EB = 4, CE = 2; whence we readily find the area = 9. Suppose PQ, RS to be the two dividing lines intersecting at right angles in O.

Take A for origin of rectangular coordinates, and AB for axis of x. The equations of AB, BC, CD, DA will be respectively $y = 0$, $x + 2y = 8$, $x - 3y = -2$, $x - y = 0$. Suppose $x - uy = v$ and $x + u^{-1}y = w$ to be the equations of PQ and RS respectively. We easily find the coordinates of P, Q, R, S, O to be

$$P_x = P_y = \frac{v}{1-u},\quad Q_x = \frac{8u+2v}{2+u},\quad Q_y = \frac{8-v}{2+u},\quad R_x = w,\quad R_y = 0,$$

$$S_x = \frac{3w - 2u^{-1}}{3+u^{-1}},\quad S_y = \frac{2+w}{3+u^{-1}},\quad O_x = \frac{uw + u^{-1}v}{u+u^{-1}},\quad O_y = \frac{w-v}{u+u^{-1}}.$$

From these results we readily find

$$\text{twice area APQB} = \frac{3v^2 - 16(1-u)v + 64(1-u)}{(1-u)(2+u)} = 9 \;\ldots\ldots\ldots\ldots (1),$$

$$\text{twice area ARSD} = \frac{w^2 + 4w - 2(1+u^{-1})}{3+u^{-1}} = 9 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\text{twice area AROP} = \frac{w^2(1-u)-2(1-u)wv+(1+u^{-1})v^2}{(1-u)(u+u^{-1})} = \frac{9}{2} \ \ldots\ (3).$$

From (1) we find $\quad v = \frac{5}{3}(1-u) \pm \frac{1}{3}\sqrt{37}\left\{(u+2)(u-1)\right\}^{\frac{1}{2}}$,

and from (2), $\qquad w = -2 \pm \sqrt{11}\,(3+u^{-1})^{\frac{1}{2}}$.

By substituting these results in (3), we shall have an equation in u from which u may be found, and thence v and w, which completely solves the question. The equation in u will be rather complicated, and I do not see how to simplify it.

3024. (Proposed by A. B. Evans, M.A.)—Find integral values of m, n, r, s that will satisfy the condition

$$m^2-n^2 = 5(m^2-r^2) = 7(m^2-s^2).$$

Solution by R. Tucker, M.A.

The given equalities reduce to

$$n^2-5r^2=-4m^2,\quad n^2-7s^2=-6m^2,\quad n^2-15r^2=-14s^2 \ \ldots\ (1,2,3).$$

It is readily seen that $n=r=s=\pm m$ satisfy all three. In (1) the convergents to $\sqrt{5}$ are $\frac{2}{1}$, $\frac{9}{4}$, $\frac{38}{17}$, &c. The even convergents satisfy the equation $n^2-5r^2=-m^2$; hence one set of answers (see Todhunter's *Algebra*, Art. 641) is $\ n=9m\pm20m,\quad r=4m\pm9m,\quad s=11m$.

Making $m = 1, 2, 3$, &c., we have

$$n=29, 58, 87, \text{\&c.},\quad r=13, 26, 39, \text{\&c.},\quad s=11, 22, 33, \text{\&c.}$$

3388. (Proposed by H. M'Coll.)—What are the respective chances that the resultant of two random forces in the same plane will be the greatest, the least, or intermediate one of the three?

Solution by Stephen Watson.

Let x, y represent the two forces, ϕ the angle between them, and z the resultant; then $\qquad x^2+y^2+2xy\cos\phi = z^2$ .... (1).

All necessary values will be taken by supposing x, y each to lie between 0 and a, and ϕ between 0 and π; thus the whole number of ways the forces can be taken is $a^2\pi$.

From (1) it is plain that z will be greater than either x or y for all values of x and y, when $\phi<\frac{1}{2}\pi$, and the chance of this is $\frac{1}{2}$.

Put now $\phi=\frac{1}{2}\pi+\theta$, then (1) becomes $x^2+y^2-2xy\sin\theta=z^2$; and if $x<y$, the condition $z>y$ gives $x>2y\sin\theta$, and therefore $\theta<\frac{1}{6}\pi$; hence doubling because y may be $<x$, the chance that z shall be the greatest of

the forces is $\frac{1}{2} - \frac{2}{a^2\pi}\int_0^{\frac{1}{6}\pi} d\theta \int_0^a dy \int_{2y\sin\theta}^y dx = \frac{2}{3} - \frac{2-\sqrt{3}}{\pi}$ ....(2).

In like manner, that z shall be the least of the forces, the chance is

$$\frac{2}{a^2\pi}\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} d\theta \int_0^a dx \int_x^{2x\sin\theta} dy = \frac{\sqrt{3}}{\pi} - \frac{1}{3} \quad\ldots\ldots\ldots\ldots(3).$$

Consequently the chance of z being intermediate between x and y is

$$1-(2)-(3) = \frac{2}{3} - \frac{2(\sqrt{3}-1)}{\pi}.$$

The chances being independent of a, hold for all values of a, and therefore when a is infinite.

3373. (Proposed by T. Morley.)—A hollow sphere composed of matter whose specific gravity is s, is filled with a liquid having the specific gravity s', and the sphere just floats in a fluid of specific gravity w. Express the ratio of the external and internal diameters of the shell in terms of the specific gravities s, s', w.

Solution by the Proposer; A. Martin; *and others.*

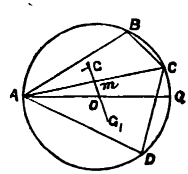
Let D, d be the external and internal diameters; then

$\frac{1}{6}\pi(D^3-d^3)$ = volume of the shell, and $\frac{1}{6}\pi d^3$ = volume of liquid it contains

therefore $\frac{1}{6}\pi(D^3-d^3)s + \frac{1}{6}\pi d^3 s' = \frac{1}{6}\pi D^3 w$, whence $\frac{D}{d} = \left(\frac{s-s'}{s-w}\right)^{\frac{1}{3}}$.

2899. (Proposed by A. Martin.)—Find (1) the average area of all the triangles that can be formed by joining three points taken at random in the circumference of a given circle, and (2) the average area of all the quadrilaterals that can be formed by joining four points taken at random in the circumference of the same circle.

Solution by Stephen Watson.



1. Let A, B, C, D be any four points in the circumference of the circle, centre O; AQ a diameter; G and G_1 the centres of gravity of the arcs ABC, ADC. Then GOG_1 is a straight line bisecting AC perpendicularly in m. Put $AO = a$, $\angle CAQ = \phi$. Then $OG = \frac{a \cdot AC}{\text{arc ABC}}$, and the sum of the areas of the triangle ABC, when B takes every point on the arc ABC, is $Am \cdot mG \cdot \text{arc ABC} = a^3\cos\phi\left\{2\cos\phi - \sin\phi(\pi-2\phi)\right\}$ ...(1).

Similarly, when B takes all positions on the arc ADC, the sum of the areas is $$a^3\cos\phi\left\{2\cos\phi+\sin\phi\,(\pi+2\phi)\right\}\ \dots\dots\dots\dots (2).$$

Also the total number of positions of B and C is $4a^2\pi^2$, and an element of the circumference at C is $d\,(2a\phi)$; hence the required average is

$$\frac{1}{4a^2\pi^2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left\{(1)+(2)\right\}2a\,d\phi=\frac{3a^2}{2\pi}.$$

2. The sum of the areas of the quadrilateral ABCD, when B and D take every position on the arcs ABC and ADC, is

$$Am\,.\,GG_1\,.\,\text{arc ABC}\,.\,\text{arc ADC}=4a^4\pi\cos^2\phi;$$

hence, in this case, the average is

$$\frac{6}{8a^3\pi^3}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}4a^4\pi\cos^2\phi\,d\,(2a\phi)=\frac{3a^2}{\pi},$$

being multiplied by 6 because the points B, C, D may be interchanged in six ways.

3256. (Proposed by Rev. J. Wolstenholme, M.A.)—If a given finite straight line be divided at random into four parts, the chance that the sum of the squares on any three of the parts shall be together less than the square on the fourth, is $6+\pi-12\log 2$.

Solution by Stephen Watson.

Let $a=$ the given line, x, y, z the three parts; then we must have $x^2+y^2+z^2<(a-x-y-z)^2$; therefore

$$z<\frac{(a-x-y)^2-x^2-y^2}{2\,(a-x-y)}\ (=z',\text{ say}),\quad y<\frac{(a-x)^2-x^2}{2\,(a-x)}\ (=y',\text{ say}),\quad x<\tfrac{1}{2}a.$$

Multiplying by 4 because any one of the four parts may be the *fourth* part, and again by 6 because the three points of division may be interchanged in six ways; also dividing by a^3 the number of ways the three points can be taken on the line, the chance required is

$$p=\frac{24}{a^3}\int_0^{\frac{1}{2}a}dx\int_0^{y'}dy\left\{\frac{(a-x-y)^2-x^2-y^2}{2\,(a-x-y)}\right\}$$

$$=3+\frac{12}{a^2}\int_0^{\frac{1}{2}a}\left\{(a-x)^2+x^2\right\}\log\frac{(a-x)^2+x^2}{2\,(a-x)^2}$$

$$=3-4\log 2-\frac{8}{a^2}\int_0^{\frac{1}{2}a}x\,dx\,\frac{x^3-(a-x)^3}{(a-x)\left\{(a-x)^2+x^2\right\}}$$

$$=3-4\log 2+\frac{8}{a^2}\int_0^{\frac{1}{2}a}dx\left\{x+\tfrac{1}{2}a-\frac{a^2}{a-x}+\frac{\frac{1}{2}a^3}{\frac{1}{2}a^2-2\,(\frac{1}{2}a-x)^2}\right\}$$

$$=6+\pi-12\log 2.$$

The above solution is on the supposition that, after taking the three points at random, we may take each one of the parts into which a is divided successively as the fourth part, thus giving four different ways to try for favourable cases for each position of the three points. If only one of the four parts, taken at random, be taken as the fourth part for each position of the three points, the chance is one-fourth of that given above.

Note on Maxima and Minima. *By* Artemas Martin.

Let it be required to determine the condition that

$$F(x) = x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + dx^{n-4} + \&c. \quad\ldots\ldots\ldots\ldots (1)$$

shall be a maximum or minimum.

Substituting $x+h$ and $x-h$ successively in (1), we obtain

$$F(x+h) = (x+h)^n + a(x+h)^{n-1} + b(x+h)^{n-2} + c(x+h)^{n-3} + \&c.\ldots (2),$$

$$F(x-h) = (x-h)^n + a(x-h)^{n-1} + b(x-h)^{n-2} + c(x-h)^{n-3} + \&c.\ldots (3),$$

where h may be any number whatever, small or large.

When $F(x)$ is a maximum, $F(x+h) - F(x)$ and $F(x-h) - F(x)$ must both be negative; but when $F(x)$ is a minimum, they must both be positive.

Expanding the right hand members of (2) and (3), and subtracting (1) from each, $F(x+h) - F(x) = Ah + Bh^2 + Ch^3 + Dh^4 + \&c. \ \ldots\ldots\ldots\ldots (4)$, where

$$A = nx^{n-1} + (n-1)ax^{n-2} + (n-2)bx^{n-3} + (n-3)cx^{n-4} + (n-4)dx^{n-5} + \&c.,$$

$$B = \frac{n(n-1)}{1.2}x^{n-2} + \frac{(n-1)(n-2)}{1.2}ax^{n-3} + \frac{(n-2)(n-3)}{1.2}bx^{n-4} + \frac{(n-3)(n-4)}{1.2}cx^{n-5} + \&c.,$$

$$C = \frac{n(n-1)(n-2)}{1.2.3}x^{n-3} + \frac{(n-1)(n-2)(n-3)}{1.2.3}ax^{n-4} + \frac{(n-2)(n-3)(n-4)}{1.2.3}bx^{n-5} + \&c.,$$

$$D = \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^{n-4} + \frac{(n-1)(n-2)(n-3)(n-4)}{1.2.3.4}ax^{n-5} + \&c.;$$

and $\quad F(x-h) - F(x) = -Ah + Bh^2 - Ch^3 + Dh^4 - \&c. \ldots\ldots\ldots\ldots (5)$,

where A, B, C, &c., have the same values as above.

Since the value of h is entirely arbitrary, it may be assumed so small that the term containing the first power of h as a factor shall be greater than all the terms containing higher powers of h. But if A be any real positive number, (4) cannot be negative, and if A be any real negative number, (5) cannot be negative; also, for positive values of A, (5) cannot be positive, and for negative values of A, (4) cannot be positive. Therefore, since A can neither be a positive nor negative number, it must be zero, and the required condition is

$$nx^{n-1} + (n-1)ax^{n-2} + (n-2)bx^{n-3} + (n-3)cx^{n-4} + \&c. = 0 \ \ldots\ldots (6),$$

which agrees with the result obtained by the theory of the Differential Calculus.

Extraction of the Roots of whole Numbers by the "Binomial Theorem." *By* Artemas Martin.

In many instances the "Binomial Theorem" may be applied with advantage in the extraction of the higher roots of the whole numbers. Thus, we have

$$(a^n \pm x)^{\frac{1}{n}} = a \pm \frac{x}{na^{n-1}} - \frac{(n-1)x^2}{1.2.n^2a^{2n-1}} \pm \frac{(n-1)(n-2)x^3}{1.2.3.n^3a^{3n-1}}$$
$$- \frac{(n-1)(2n-1)(3n-1)x^4}{1.2.3.4.n^4a^{4n-1}} + \&c.$$

Ex. 1.—Required the square root of 26.
Making $a=5$, $x=1$, $n=2$, we have

$$(26)^{\frac{1}{2}} = 5 + \tfrac{1}{10} - \tfrac{1}{1000} + \tfrac{1}{20000} - \tfrac{1}{2000000} + \&c., = 5.09909+.$$

Ex. 2. Required the square root of 216.

$$(216)^{\frac{1}{2}} = 3(24)^{\frac{1}{2}} = 3(5^2-1)^{\frac{1}{2}}.$$

Making $a=5$, $n=2$, $x=1$, we have

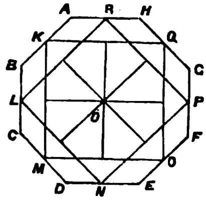
$$(24)^{\frac{1}{2}} = 5 - \tfrac{1}{10} - \tfrac{1}{1000} - \tfrac{1}{20000} - \tfrac{1}{2000000} - \&c.,$$
$$= 4{\cdot}898974+; \text{ therefore } (216)^{\frac{1}{2}} = 14{\cdot}69692+.$$

Ex. 3. Required the fifth root of 30.

$$(30)^{\frac{1}{5}} = (32-2)^{\frac{1}{5}} = 2 - \frac{1}{5.2^3} - \frac{1}{5^2.2^6} - \frac{3}{5^3.2^{10}} - \&c.$$
$$= 1{\cdot}974351+.$$

3287. (Proposed by W. H. H. Hudson, M.A.)—Two quadrilaterals are formed from an octagon by joining the middle points of alternate sides. Prove that the lines joining the middle points of the opposite sides of these quadrilaterals pass though a point.

Solutions by the Rev. J. L. Kitchin, M.A.; Rev. J. Wolstenholme, M.A.; *the* Proposer; *and others.*



1. Take a regular octagon ABCDEFGH. Then KMOQ, RLNP are the quadrilaterals formed by joining alternate middle points of the sides.

They are obviously equal squares whose centre coincides with the centre of the circumscribing circle.

The four lines joining the middle points of the opposite sides of the two quadrilaterals obviously pass through the same point.

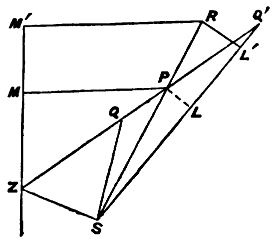
2. Or, generally, for any octagon. Place equal heavy particles at the corners of the octagon. Proceed to find their centre of gravity. Combine them two and two, we

obtain double particles at the corners of one of the quadrilaterals, found by joining the middle points of alternate sides; the centre of gravity of these is the middle point of the lines joining the middle points of opposite sides. Similarly we may obtain, combining them differently, another quadrilateral; and since there is but one centre of gravity, the lines joining the middle points of opposite sides of both these quadrilaterals pass through a point which bisects each of them.

3061. (Proposed by C. TAYLOR, M.A.)—If a straight line envelope a conic, the locus of its pole with respect to a conic will be a conic. Prove by elementary geometry that if two of the conics have the same focus and directrix, the third will have the same focus and directrix, and the eccentricities of the three will be proportionals.

Solution by R. W. GENESE, B.A.

If QQ′ be a chord of a conic A touching at P a conic B having the same focus S and directrix ZM, then SZ (in fig.) is the external bisector of the angle QSQ′, and therefore SP is the internal bisector. Draw PL parallel to ZS, or perpendicular to SP.



SL : PM = eccentricity of A(e),

and SP : PM = " B(e');

therefore SL : SP = $e : e'$.

Let R be the the pole of QQ′ with respect to A; RM′, RL′ perpendicular to ZM, SQ′ respectively.

Then $SL' = e\,RM'$ and $SR = \frac{e}{e'}SL'$, therefore $SR = \frac{e^2}{e'}RM'$;

that is, the locus of R is a conic having the same focus and directrix as A and B, and eccentricity a third proportional to e' and e.

3323. (Proposed by the Rev. E. HILL, M.A.)—A straight pipe, 40 feet long, 6 inches in diameter, closed at the top, and full of ice, is inverted in a barrel a yard in diameter. Given the specific gravity of ice ·9, and the height of the water barometer 30 feet; show that, when the ice melts, the water will rise 2 inches in the barrel.

Solution by the REV. J. L. KITCHIN, M.A.

The ice when melted is equal to a column of water $= 40 \times \cdot 9 = 36$ feet. We have to account for 6 feet of this.

Let $x =$ the number of inches the water rises in the barrel; then $72-x$ is the height of column which escapes into the barrel;

therefore $\pi . x . 324 - 9\pi . x = (72-x)\, 9\pi$, whence $x = 2$.

3318. (Proposed by R. TUCKER, M.A.)—Inscribe the maximum rectangle in a lemniscate.

Solution by the PROPOSER.

If the rectangle be inscribed in one loop of the curve, we have, (ρ, θ), (ρ', θ') being the coordinates of two angular points,

$$\text{the area} = 2u = 2\,(\rho\cos\theta - \rho'\cos\theta')\,\rho\sin\theta \quad\text{.................(1)},$$

and $$\rho\sin\theta = \rho'\sin\theta' \quad\text{.................(2)}.$$

Hence $$\text{area} = \tfrac{1}{4}a^2\,(\sin 4\theta - \sin 4\theta').$$

For a maximum, $$\cos 4\theta\, d\theta = \cos 4\theta'\, d\theta' \quad\text{.................(3)}.$$

Again, (2) gives us $\cos 2\theta - \cos 2\theta' = \cos^2 2\theta - \cos^2 2\theta'$,

i.e., $$\cos 2\theta = \cos 2\theta', \quad\text{or}\quad \cos 2\theta + \cos 2\theta' = 1 \quad\text{.................(4)}.$$

[This last equation gives us $\rho^2 + \rho'^2 = a^2$.]

Differentiating (4), $$\sin 2\theta\, d\theta + \sin 2\theta'\, d\theta' = 0 \quad\text{.................(5)}.$$

Elimiminating the differentials, we obtain

$$\sin 2\theta\cos 4\theta' + \sin 2\theta'\cos 4\theta = 0,$$

or $$\sin 2\theta + \sin 2\theta' = 2\sin 2\theta\sin 2\theta'\,(\sin 2\theta + \sin 2\theta');$$

and $$\sin 2\theta + \sin 2\theta' = 0, \quad\text{or}\quad \sin 2\theta\sin 2\theta' = \tfrac{1}{2} \quad\text{.................(6)}.$$

Taking the equations $\cos 2\theta' = 2\sin^2\theta$, $\sin 2\theta' = \dfrac{1}{2\sin 2\theta}$,

we have, if $x = 2\sin^2\theta$, $4x^4 - 8x^3 - 4x^2 + 8x = 1$;

whence $x(x-1) = 1 \pm \dfrac{\sqrt{5}}{2}$, and $\sin^2\theta = \dfrac{1 \pm \sqrt{5}\,.\,\sqrt{(\sqrt{5} \pm 2)}}{4}$;

the only admissible values are

$$\sin^2\theta = \frac{1 - \sqrt{(5 - 2\sqrt{5})}}{4}, \quad \therefore\ \sin\theta = \sqrt{(\cdot065865)} = \cdot25664,$$

which gives $\theta = 15°$ very nearly,

and $$\sin^2\theta = \frac{1 + \sqrt{(5 - 2\sqrt{5})}}{4}, \quad \therefore\ \sin\theta = \sqrt{(\cdot431635)} = \cdot65698,$$

which gives $\theta = 41° 4'$ approximately.

Hence, in the maximum position, we have

$$\theta = 15° \text{ nearly}, \quad\text{and}\quad \theta' = 41° 4' \text{ nearly}.$$

Next, if the rectangle have its opposite angles on different branches of the curve, it is easily seen that the area $= a^2\sin^4\theta$, which is a maximum when $\theta = \frac{1}{3}\pi$.

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